CHAPTER 2 Complex Numbers

Section-A JEE Advanced/ IIT-JEE

A Fill in the Blanks

1. If the expression (1987 - 2 Marks)

$$\frac{\left[\sin\left(\frac{x}{2}\right)+\cos\left(\frac{x}{2}\right)+i\tan(x)\right]}{\left[1+2\,i\,\sin\left(\frac{x}{2}\right)\right]}$$

is real, then the set of all possible values of x is ....

2. For any two complex numbers z_1, z_2 and any real number a and b. *(1988 - 2 Marks)*
$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 =$ ....

3. If a, b, c, are the numbers between 0 and 1 such that the points $z_1 = a + i, z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then $a =$ ....and $b =$ .... *(1989 - 2 Marks)*

4. *ABCD* is a rhombus. Its diagonals *AC* and *BD* intersect at the point *M* and satisfy *BD* = 2*AC*. If the points *D* and *M* represent the complex numbers 1 + *i* and 2 - *i* respectively, then A represents the complex number ....or.......... *(1993 - 2 Marks)*

5. Suppose Z_1, Z_2, Z_3 are the vertices of an equilateral triangle inscribed in the circle $|Z| = 2$. If $Z_1 = 1 + i\sqrt{3}$ then $Z_2 =$ ...., $Z_3 =$ .... *(1994 - 2 Marks)*

6. The value of the expression
$1\cdot(2-\omega)(2-\omega^2)+2\cdot(3-\omega)(3-\omega^2)+....+(n-1).(n-\omega)(n-\omega^2)$,
where ω is an imaginary cube root of unity, is.....
(1996 - 2 Marks)

B True / False

1. For complex number $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$, if $x_1 \le x_2$ and $y_1 \le y_2$. Then for all complex numbers *z* with $1 \cap z$, we have $\frac{1-z}{1+z} \cap 0$. *(1981 - 2 Marks)*

2. If the complex numbers, Z_1, Z_2 and Z_3 represent the vertices of an equilateral triangle such that
$|Z_1| = |Z_2| = |Z_3|$ then $Z_1 + Z_2 + Z_3 = 0$. *(1984 - 1 Mark)*

3. If three complex numbers are in A.P. then they lie on a circle in the complex plane. *(1985 - 1 Mark)*

4. The cube roots of unity when represented on Argand diagram form the vertices of an equilateral triangle.
(1988 - 1 Mark)

C MCQs with One Correct Answer

1. If the cube roots of unity are 1, ω, ω^2, then the roots of the equation $(x-1)^3 + 8 = 0$ are *(1979)*
(a) $-1, 1+2\omega, 1+2\omega^2$ (b) $-1, 1-2\omega, 1-2\omega^2$
(c) $-1, -1, -1$ (d) None of these

2. The smallest positive integer n for which *(1980)*
$\left(\frac{1+i}{1-i}\right)^n = 1$ is
(a) $n = 8$ (b) $n = 16$
(c) $n = 12$ (d) none of these

3. The complex numbers $z = x + iy$ which satisfy the equation
$\left|\frac{z-5i}{z+5i}\right| = 1$ lie on *(1981 - 2 Marks)*
(a) the x-axis
(b) the straight line $y = 5$
(c) a circle passing through the origin
(d) none of these

4. If $z = \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$, then *(1982 - 2 Marks)*
(a) $\text{Re}(z) = 0$ (b) $\text{Im}(z) = 0$
(c) $\text{Re}(z) > 0, \text{Im}(z) > 0$ (d) $\text{Re}(z) > 0, \text{Im}(z) < 0$

5. The inequality $|z-4| < |z-2|$ represents the region given by *(1982 - 2 Marks)*
(a) $\text{Re}(z) \ge 0$ (b) $\text{Re}(z) < 0$
(c) $\text{Re}(z) > 0$ (d) none of these

6. If $z = x + iy$ and $\omega = (1-iz)/(z-i)$, then $|\omega| = 1$ implies that, in the complex plane, *(1983 - 1 Mark)*
(a) *z* lies on the imaginary axis
(b) *z* lies on the real axis
(c) *z* lies on the unit circle
(d) None of these

7. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if *(1983 - 1 Mark)*
(a) $z_1+z_4=z_2+z_3$ (b) $z_1+z_3=z_2+z_4$
(c) $z_1+z_2=z_3+z_4$ (d) None of these

8. If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that $c=(1-r)a+rb$ and $w=(1-r)u+rv$, where r is a complex number, then the two triangles *(1985 - 2 Marks)*
(a) have the same area (b) are similar
(c) are congruent (d) none of these

9. If $\omega\ (\neq 1)$ is a cube root of unity and $(1+\omega)^7=A+B\omega$ then A and B are respectively *(1995S)*
(a) 0, 1 (b) 1, 1 (c) 1, 0 (d) $-1, 1$

10. Let z and ω be two non zero complex numbers such that $|z|=|\omega|$ and $\text{Arg}\,z+\text{Arg}\,\omega=\pi$, then z equals *(1995S)*
(a) ω (b) $-\omega$ (c) $\overline{\omega}$ (d) $-\overline{\omega}$

11. Let z and ω be two complex numbers such that $|z|\le 1$, $|\omega|\le 1$ and $|z+i\omega|=|z-i\overline{\omega}|=2$ then z equals *(1995S)*
(a) 1 or i (b) i or $-i$ (c) 1 or -1 (d) i or -1

12. For positive integers n_1, n_2 the value of the expression $(1+i)^{n_1}+(1+i^3)^{n_1}+(1+i^5)^{n_2}+(1+i^7)^{n_2}$, where $i=\sqrt{-1}$ is a real number if and only if *(1996 - 1 Marks)*
(a) $n_1=n_2+1$ (b) $n_1=n_2-1$
(c) $n_1=n_2$ (d) $n_1>0, n_2>0$

13. If $i=\sqrt{-1}$, then $4+5\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{334}+3\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{365}$ is equal to *(1999 - 2 Marks)*
(a) $1-i\sqrt{3}$ (b) $-1+i\sqrt{3}$ (c) $i\sqrt{3}$ (d) $-i\sqrt{3}$

14. If $\arg(z)<0$, then $\arg(-z)-\arg(z)=$ *(2000S)*
(a) π (b) $-\pi$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

15. If z_1, z_2 and z_3 are complex numbers such that *(2000S)*
$|z_1|=|z_2|=|z_3|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$, then $|z_1+z_2+z_3|$ is
(a) equal to 1 (b) less than 1
(c) greater than 3 (d) equal to 3

16. Let z_1 and z_2 be n^{th} roots of unity which subtend a right angle at the origin. Then n must be of the form *(2001S)*
(a) $4k+1$ (b) $4k+2$ (c) $4k+3$ (d) $4k$

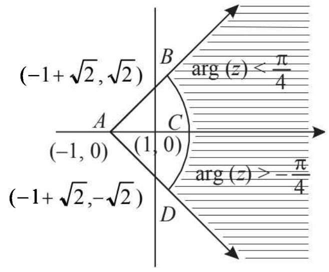
17. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1-z_3}{z_2-z_3}=\frac{1-i\sqrt{3}}{2}$ are the vertices of a triangle which is *(2001S)*
(a) of area zero
(b) right-angled isosceles
(c) equilateral
(d) obtuse-angled isosceles

18. For all complex numbers z_1, z_2 satisfying $|z_1|=12$ and $|z_2-3-4i|=5$, the minimum value of $|z_1-z_2|$ is *(2002S)*
(a) 0 (b) 2 (c) 7 (d) 17

19. If $|z|=1$ and $\omega=\frac{z-1}{z+1}$ (where $z\neq -1$), then $\text{Re}(\omega)$ is *(2003S)*
(a) 0 (b) $-\frac{1}{|z+1|^2}$
(c) $\left|\frac{z}{z+1}\right|\cdot\frac{1}{|z+1|^2}$ (d) $\frac{\sqrt{2}}{|z+1|^2}$

20. If $\omega\ (\neq 1)$ be a cube root of unity and $(1+\omega^2)^n=(1+\omega^4)^n$, then the least positive value of n is *(2004S)*
(a) 2 (b) 3 (c) 5 (d) 6

21. The locus of z which lies in shaded region (excluding the boundaries) is best represented by *(2005S)*



(a) $z:|z+1|>2$ and $|\arg(z+1)|<\pi/4$
(b) $z:|z-1|>2$ and $|\arg(z-1)|<\pi/4$
(c) $z:|z+1|<2$ and $|\arg(z+1)|<\pi/2$
(d) $z:|z-1|<2$ and $|\arg(z+1)|<\pi/2$

22. a, b, c are integers, not all simultaneously equal and ω is cube root of unity ($\omega\neq 1$), then minimum value of $|a+b\omega+c\omega^2|$ is *(2005S)*
(a) 0 (b) 1 (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

23. Let $\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$, then the value of the det.
$\begin{vmatrix}1 & 1 & 1\\ 1 & -1-\omega^2 & \omega^2\\ 1 & \omega^2 & \omega^4\end{vmatrix}$ is *(2002 - 2 Marks)*
(a) 3ω (b) $3\omega(\omega-1)$
(c) $3\omega^2$ (d) $3\omega(1-\omega)$

24. If $\frac{w-\overline{w}z}{1-z}$ is purely real where $w=\alpha+i\beta$, $\beta\neq 0$ and $z\neq 1$, then the set of the values of z is *(2006 - 3M, –1)*
(a) $\{z:|z|=1\}$ (b) $\{z:z=\overline{z}\}$
(c) $\{z:z\neq 1\}$ (d) $\{z:|z|=1, z\neq 1\}$

25. A man walks a distance of 3 units from the origin towards the north-east (N 45° E) direction. From there, he walks a distance of 4 units towards the north-west (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is *(2007 -3 marks)*
(a) $3e^{i\pi/4}+4i$ (b) $(3-4i)e^{i\pi/4}$
(c) $(4+3i)e^{i\pi/4}$ (d) $(3+4i)e^{i\pi/4}$

26. If $|z|=1$ and $z\neq\pm1$, then all the values of $\frac{z}{1-z^2}$ lie on
(a) a line not passing through the origin *(2007 -3 marks)*
(b) $|z|=\sqrt{2}$
(c) the x-axis
(d) the y-axis

27. A particle P starts from the point $z_0=1+2i$, where $i=\sqrt{-1}$. It moves horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i}+\hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point z_2. The point z_2 is given by *(2008)*
(a) $6+7i$ (b) $-7+6i$
(c) $7+6i$ (d) $-6+7i$

28. Let $z=\cos\theta+i\sin\theta$. Then the value of $\sum_{m=1}^{15}\text{Im}(z^{2m-1})$ at $\theta=2°$ is *(2009)*
(a) $\frac{1}{\sin 2°}$ (b) $\frac{1}{3\sin 2°}$ (c) $\frac{1}{2\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

29. Let $z=x+iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation : $z\bar{z}^3+\bar{z}z^3=350$ is *(2009)*
(a) 48 (b) 32 (c) 40 (d) 80

30. Let z be a complex number such that the imaginary part of z is non-zero and $a=z^2+z+1$ is real. Then a cannot take the value *(2012)*
(a) -1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

31. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x-x_0)^2+(y-y_0)^2=r^2$ and $(x-x_0)^2+(y-y_0)^2=4r^2$, respectively. If $z_0=x_0+iy_0$ satisfies the equation $2|z_0|^2=r^2+2$, then $|\alpha|=$ *(JEE Adv. 2013)*
(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{3}$

D MCQs with One or More than One Correct

1. If $z_1=a+ib$ and $z_2=c+id$ are complex numbers such that $|z_1|=|z_2|=1$ and $\text{Re}(z_1\bar{z}_2)=0$, then the pair of complex numbers $w_1=a+ic$ and $w_2=b+id$ satisfies – *(1985 - 2 Marks)*
(a) $|w_1|=1$ (b) $|w_2|=1$
(c) $\text{Re}(w_1\bar{w}_2)=0$ (d) none of these

2. Let z_1 and z_2 be complex numbers such that $z_1\neq z_2$ and $|z_1|=|z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1+z_2}{z_1-z_2}$ may be *(1986 - 2 Marks)*
(a) zero (b) real and positive
(c) real and negative (d) purely imaginary
(e) none of these.

3. If z_1 and z_2 are two nonzero complex numbers such that $|z_1+z_2|=|z_1|+|z_2|$, then Arg z_1 – Arg z_2 is equal to *(1987 - 2 Marks)*
(a) $-\pi$ (b) $-\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{2}$
(e) π

4. The value of $\sum_{k=1}^{6}(\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7})$ is *(1987 - 2 Marks)*
(a) -1 (b) 0 (c) $-i$ (d) i
(e) None

5. If ω is an imaginary cube root of unity, then $(1+\omega-\omega^2)^7$ equals *(1998 - 2 Marks)*
(a) 128ω (b) -128ω (c) $128\omega^2$ (d) $-128\omega^2$

6. The value of the sum $\sum_{n=1}^{13}(i^n+i^{n+1})$, where $i=\sqrt{-1}$, equals *(1998 - 2 Marks)*
(a) i (b) $i-1$ (c) $-i$ (d) 0

7. If $\begin{vmatrix}6i & -3i & 1\\ 4 & 3i & -1\\ 20 & 3 & i\end{vmatrix}=x+iy$, then *(1998 - 2 Marks)*
(a) $x=3, y=2$ (b) $x=1, y=3$
(c) $x=0, y=3$ (d) $x=0, y=0$

8. Let z_1 and z_2 be two distinct complex numbers and let $z=(1-t)z_1+tz_2$ for some real number t with $0<t<1$. If Arg (w) denotes the principal argument of a non-zero complex number w, then *(2010)*
(a) $|z-z_1|+|z-z_2|=|z_1-z_2|$
(b) Arg $(z-z_1)=$ Arg $(z-z_2)$
(c) $\begin{vmatrix}z-z_1 & \bar{z}-\bar{z}_1\\ z_2-z_1 & \bar{z}_2-\bar{z}_1\end{vmatrix}$
(d) Arg $(z-z_1)=$ Arg (z_2-z_1)

9. Let $w = \frac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1, 2, 3, ...\}$. Further $H_1 = \left\{z \in \mathbb{C} : \text{Re}\, z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in \mathbb{C} : \text{Re}\, z < \frac{-1}{2}\right\}$, where c is the set of all complex numbers. If $z_1 \in P \cap H_1, z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1 O z_2 =$ *(JEE Adv. 2013)*

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

10. Let a, b $\in \mathbb{R}$ and $a^2 + b^2 \neq 0$.

Suppose $S = \left\{Z \in C : Z = \frac{1}{a+ibt}, + \in \mathbb{R}, t \neq 0\right\}$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in S$, then (x, y) lies on *(JEE Adv. 2016)*

(a) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0$, $b \neq 0$

(b) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(c) the x-axis for $a \neq 0, b = 0$

(d) the y-axis for $a = 0, b \neq 0$

E Subjective Problems

1. Express $\frac{1}{1-\cos\theta + 2i\sin\theta}$ in the form $x + iy$. *(1978)*

2. If $x = a + b, y = a\gamma + b\beta$ and $z = a\beta + b\gamma$ where γ and β are the complex cube roots of unity, show that $xyz = a^3 + b^3$. *(1978)*

3. If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, prove that $(x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$. *(1979)*

4. Find the real values of x and y for which the following equation is satisfied $\frac{(1+i)x-2i}{3+i} + \frac{(2-3i)y+i}{3-i} = i$ *(1980)*

5. Let the complex number z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$. *(1981 - 4 Marks)*

6. Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only if $z_1^2 + z_2^2 - z_1 z_2 = 0$. *(1983 - 3 Marks)*

7. If $1, a_1, a_2 ......, a_{n-1}$ are the n roots of unity, then show that $(1-a_1)(1-a_2)(1-a_3)....(1-a_{n-1}) = n$ *(1984 - 2 Marks)*

8. Show that the area of the triangle on the Argand diagram formed by the complex numbers z, iz and $z + iz$ is $\frac{1}{2}|z|^2$. *(1986 - 2½ Marks)*

9. Let $Z_1 = 10 + 6i$ and $Z_2 = 4 + 6i$. If Z is any complex number such that the argument of $\frac{(Z-Z_1)}{(Z-Z_2)}$ is $\frac{\pi}{4}$, then prove that $|Z - 7 - 9i| = 3\sqrt{2}$. *(1990 - 4 Marks)*

10. If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$. *(1995 - 5 Marks)*

11. If $|Z| \leq 1, |W| \leq 1$, show that

$$|Z - W|^2 \leq (|Z| - |W|)^2 + (Arg\, Z - Arg\, W)^2$$

(1995 - 5 Marks)

12. Find all non-zero complex numbers Z satisfying $\overline{Z} = iZ^2$. *(1996 - 2 Marks)*

13. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, prove that

$p^2 = 4q\cos^2\left(\frac{\alpha}{2}\right)$. *(1997 - 5 Marks)*

14. For complex numbers z and w, prove that $|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or $z\overline{w} = 1$. *(1999 - 10 Marks)*

15. Let a complex number α, $\alpha \neq 1$, be a root of the equation $z^{p+q} - z^p - z^q + 1 = 0$, where p, q are distinct primes. Show that either $1 + \alpha + \alpha^2 + ... + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + ... + \alpha^{q-1} = 0$, but not both together. *(2002 - 5 Marks)*

16. If z_1 and z_2 are two complex numbers such taht $|z_1| < 1 < |z_2|$ then prove that $\left|\frac{1 - z_1\overline{z}_2}{z_1 - z_2}\right| < 1$. *(2003 - 2 Marks)*

17. Prove that there exists no complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$. *(2003 - 2 Marks)*

18. Find the centre and radius of circle given by

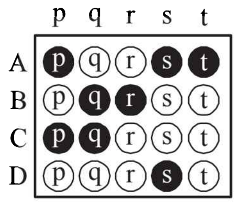
$$\left|\frac{z-\alpha}{z-\beta}\right| = k, k \neq 1$$

where, $z = x + iy$, $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ *(2004 - 2 Marks)*

19. If one the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}\, i$. Find the other vertices of the square. *(2005 - 4 Marks)*

F Match the Following

DIRECTIONS (Q. 1 and 2) : *Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :*
If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



1. $z \neq 0$ is a complex number *(1992 - 2 Marks)*

Column I	Column II
(A) $\text{Re}\, z = 0$	(p) $\text{Re}\, z^2 = 0$
(B) $\text{Arg}\, z = \dfrac{\pi}{4}$	(q) $\text{Im}\, z^2 = 0$
	(r) $\text{Re}\, z^2 = \text{Im}\, z^2$

2. Match the statements in **Column I** with those in **Column II**. *(2010)*
[Note : Here z takes values in the complex plane and Im z and Re z denote , respectively, the imaginary part and the real part of z.]

Column I	Column II
(A) The set of points z satisfying $|z - i|z|| = |z + i|z||$ is contained in or equal to	(p) an ellipse with eccentricity $\dfrac{4}{5}$
(B) The set of points z satisfying $|z+4| + |z-4| = 10$ is contained in or equal to	(q) the set of points z satisfying $\text{Im}\, z = 0$
	(r) the set of points z satisfying $|\text{Im}\, z| \le 1$
(C) If $|w| = 2$, then the set of points $z = w - \dfrac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $|\text{Re}\, z| < 2$
(D) If $|w| = 1$, then the set of points $z = w + \dfrac{1}{w}$ is contained in or equal to.	(t) the set of points z satisfying $|z| \le 3$

DIRECTIONS (Q. 3) : *Following question has matching lists. The codes for the list have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.*

3. Let $z_k = \cos\left(\dfrac{2k\pi}{10}\right) + i \sin\left(\dfrac{2k\pi}{10}\right); k = 1, 2, ..., 9.$ *(JEE Adv. 2014)*

List-I	List-II
P. For each z_k there exists as z_j such that $z_k \cdot z_j = 1$	**1.** True
Q. There exists a $k \in \{1, 2, ..., 9\}$ such that $z_1 . z = z_k$ has no solution z in the set of complex numbers	**2.** False
R. $\dfrac{|1-z_1||1-z_2|....|1-z_9|}{10}$ equals	**3.** 1
S. $1 - \displaystyle\sum_{k=1}^{9} \cos\left(\dfrac{2k\pi}{10}\right)$ equals	**4.** 2

	P	Q	R	S
(a)	1	2	4	3
(b)	2	1	3	4
(c)	1	2	3	4
(d)	2	1	4	3

G Comprehension Based Questions

PASSAGE-1

Let A, B, C be three sets of complex numbers as defined below

$A = \{z : \text{Im } z \geq 1\}$

$B = \{z : |z - 2 - i| = 3\}$

$C = \{z : \text{Re}((1 - i)z) = \sqrt{2}\}$

1. The number of elements in the set $A \cap B \cap C$ is *(2008)*
(a) 0 (b) 1 (c) 2 (d) ∞

2. Let z be any point in $A \cap B \cap C$.
Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between *(2008)*
(a) 25 and 29 (b) 30 and 34
(c) 35 and 39 (d) 40 and 44

3. Let z be any point $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between
(a) –6 and 3 (b) –3 and 6 *(2008)*
(c) –6 and 6 (d) –3 and 9

PASSAGE-2

Let $S = S_1 \cap S_2 \cap S_3$, where

$$S_1 = \{z \in \mathbb{C} : |z| < 4\},\ S_2 = \left\{z \in \mathbb{C} : \text{Im}\left[\frac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i}\right] > 0\right\}$$

and $S_3 = \{z \in \mathbb{C} : \text{Re}\, z > 0\}$.

4. Area of $S =$ *(JEE Adv. 2013)*
(a) $\frac{10\pi}{3}$ (b) $\frac{20\pi}{3}$ (c) $\frac{16\pi}{3}$ (d) $\frac{32\pi}{3}$

5. $\min\limits_{z \in S} |1 - 3i - z| =$ *(JEE Adv. 2013)*
(a) $\frac{2 - \sqrt{3}}{2}$ (b) $\frac{2 + \sqrt{3}}{2}$
(c) $\frac{3 - \sqrt{3}}{2}$ (d) $\frac{3 + \sqrt{3}}{2}$

I Integer Value Correct Type

1. If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is *(2011)*

2. Let $\omega = e^{\frac{i\pi}{3}}$, and a, b, c, x, y, z be non-zero complex numbers such that *(2011)*
$a + b + c = x$
$a + b\omega + c\omega^2 = y$
$a + b\omega^2 + c\omega = z$
Then the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is

3. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression $\frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3} |\alpha_{4k-1} - \alpha_{4k-2}|}$ is *(JEE Adv. 2015)*

Section-B JEE Main / AIEEE

1. z and w are two nonzero complex numbers such that $|z| = |w|$ and Arg z + Arg w = π then z equals **[2002]**
(a) $\overline{\omega}$ (b) $-\overline{\omega}$ (c) ω (d) $-\omega$

2. If $|z - 4| < |z - 2|$, its solution is given by **[2002]**
(a) Re(z) > 0 (b) Re(z) < 0
(c) Re(z) > 3 (d) Re(z) > 2

3. The locus of the centre of a circle which touches the circle $|z - z_1| = a$ and $|z - z_2| = b$ externally (z, z_1 & z_2 are complex numbers) will be **[2002]**
(a) an ellipse (b) a hyperbola
(c) a circle (d) none of these

4. If z and ω are two non-zero complex numbers such that $|z\omega| = 1$ and $Arg(z) - Arg(\omega) = \frac{\pi}{2}$, then $\bar{z}\omega$ is equal to **[2003]**
(a) $-i$ (b) 1 (c) -1 (d) i

5. Let Z_1 and Z_2 be two roots of the equation $Z^2 + aZ + b = 0$, Z being complex. Further, assume that the origin, Z_1 and Z_2 form an equilateral triangle. Then **[2003]**
(a) $a^2 = 4b$ (b) $a^2 = b$ (c) $a^2 = 2b$ (d) $a^2 = 3b$

6. If $\left(\frac{1+i}{1-i}\right)^x = 1$ then **[2003]**
(a) $x = 2n+1$, where n is any positive integer
(b) $x = 4n$, where n is any positive integer
(c) $x = 2n$, where n is any positive integer
(d) $x = 4n+1$, where n is any positive integer.

7. Let z and w be complex numbers such that $\bar{z} + i\bar{w} = 0$ and arg $zw = \pi$. Then arg z equals **[2004]**
(a) $\frac{5\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{4}$

8. If $z = x - i\,y$ and $z^{\frac{1}{3}} = p + iq$, then $\left(\frac{x}{p} + \frac{y}{q}\right) \Big/ (p^2 + q^2)$ is equal to **[2004]**
(a) –2 (b) –1 (c) 2 (d) 1

9. If $|z^2 - 1| = |z|^2 + 1$, then z lies on **[2004]**
(a) an ellipse (b) the imaginary axis
(c) a circle (d) the real axis

10. If the cube roots of unity are $1, \omega, \omega^2$ then the roots of the equation $(x-1)^3 + 8 = 0$, are **[2005]**
(a) $-1, -1+2\omega, -1-2\omega^2$
(b) $-1, -1, -1$
(c) $-1, 1-2\omega, 1-2\omega^2$
(d) $-1, 1+2\omega, 1+2\omega^2$

11. If z_1 and z_2 are two non- zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg z_1 - \arg z_2$ is equal to **[2005]**
(a) $\frac{\pi}{2}$ (b) $-\pi$ (c) 0 (d) $\frac{-\pi}{2}$

12. If $\omega = \frac{z}{z - \frac{1}{3}i}$ and $|\omega| = 1$, then z lies on **[2005]**
(a) an ellipse (b) a circle
(c) a straight line (d) a parabola

13. The value of $\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$ is **[2006]**
(a) i (b) 1 (c) -1 (d) $-i$

14. If $z^2 + z + 1 = 0$, where z is complex number, then the value of
$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \ldots\ldots\ldots + \left(z^6 + \frac{1}{z^6}\right)^2$ is **[2006]**
(a) 18 (b) 54
(c) 6 (d) 12

15. If $|z + 4| \le 3$, then the maximum value of $|z+1|$ is **[2007]**
(a) 6 (b) 0 (c) 4 (d) 10

16. The conjugate of a complex number is $\frac{1}{i-1}$ then that complex number is **[2008]**
(a) $\frac{-1}{i-1}$ (b) $\frac{1}{i+1}$ (c) $\frac{-1}{i+1}$ (d) $\frac{1}{i-1}$

17. Let R be the real line. Consider the following subsets of the plane $R \times R$:
$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$
$T = \{(x, y) : x - y \text{ is an integer}\}$,
Which one of the following is true? **[2008]**
(a) Neither S nor T is an equivalence relation on R
(b) Both S and T are equivalence relation on R
(c) S is an equivalence relation on R but T is not
(d) T is an equivalence relation on R but S is not

18. The number of complex numbers z such that $|z-1| = |z+1| = |z-i|$ equals **[2010]**
(a) 1 (b) 2 (c) ∞ (d) 0

19. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re $z = 1$, then it is necessary that : **[2011]**
(a) $\beta \in (-1, 0)$ (b) $|\beta| = 1$
(c) $\beta \in (1, \infty)$ (d) $\beta \in (0, 1)$

20. If $\omega(\neq 1)$ is a cube root of unity, and $(1+\omega)^7 = A + B\omega$. Then (A, B) equals **[2011]**
(a) (1, 1) (b) (1, 0)
(c) (–1, 1) (d) (0, 1)

21. If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies : **[2012]**
(a) either on the real axis or on a circle passing through the origin.
(b) on a circle with centre at the origin
(c) either on the real axis or on a circle not passing through the origin.
(d) on the imaginary axis.

22. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals: **[JEE M 2013]**
(a) $-\theta$ (b) $\frac{\pi}{2} - \theta$ (c) θ (d) $\pi - \theta$

23. If z is a complex number such that $|z| \ge 2$, then the minimum value of $\left|z + \frac{1}{2}\right|$: **[JEE M 2014]**
(a) is strictly greater than $\frac{5}{2}$
(b) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$
(c) is equal to $\frac{5}{2}$
(d) lie in the interval (1, 2)

24. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a: **[JEE M 2015]**

(a) circle of radius 2.

(b) circle of radius $\sqrt{2}$.

(c) straight line parallel to x-axis

(d) straight line parallel to y-axis.

25. A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is: **[JEE M 2016]**

(a) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (b) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

2 Complex Numbers

Section-A : JEE Advanced/ IIT-JEE

A 1. $2n\pi,\ n\pi+\frac{\pi}{4}$ 2. $(a^2+b^2)(|z_1|^2+|z_2|^2)$ 3. $2-\sqrt{3},\ 2-\sqrt{3}$

4. $3-\frac{i}{2}$ or $1-\frac{3}{2}i$ 5. $-2,\ 1-i\sqrt{3}$ 6. $\frac{1}{4}\text{n}(\text{n}-1)(\text{n}^2+3\text{n}+4)$

B 1. T 2. T 3. F 4. T

C 1. (b) 2. (d) 3. (a) 4. (b) 5. (d) 6. (b)
7. (b) 8. (b) 9. (b) 10. (d) 11. (c)
12. (d) 13. (c) 14. (a) 15. (a) 16. (d) 17. (c)
18. (b) 19. (a) 20. (b) 21. (a) 22. (b) 23. (b)
24. (d) 25. (d) 26. (d) 27. (d) 28. (d) 29. (a)
30. (d) 31. (c)

D 1. (a, b, c) 2. (a, d) 3. (c) 4. (d) 5. (d) 6. (b)
7. (d) 8. (a, c, d) 9. (c, d) 10. (a, c, d)

E 1. $\left(\frac{1}{5+3\cos\theta}\right)+\left(\frac{-2\cot\theta/2}{5+3\cos\theta}\right)i$ 4. $x=3, y=-1$ 12. $i,\ \frac{\pm\sqrt{3}}{2}-\frac{i}{2}$ 18. centre $=\frac{\alpha-k^2\beta}{1-k^2}$, radius $=\frac{k}{|1-k^2|}|\alpha-\beta|$

19. $(1-\sqrt{3})+i,\ -i\sqrt{3},\ (\sqrt{3}+1)-i$

F 1. (A) - q ; (B) - p 2. (A) - q, r ; (B) - p; (C) - p, s, t; (D) - q, r, s, t 3. (c)

G 1. (b) 2. (c) 3. (d) 4. (b) 5. (c)

I 1. 5 2. 3 3. 4

Section-B : JEE Main/ AIEEE

1. (b) 2. (c) 3. (b) 4. (a) 5. (d) 6. (b)
7. (c) 8. (a) 9. (b) 10. (c) 11. (c) 12. (c)
13. (d) 14. (d) 15. (a) 16. (c) 17. (d) 18. (a)
19. (c) 20. (a) 21. (a) 22. (c) 23. (b) 24. (a)
25. (b)

Section-A JEE Advanced/ IIT-JEE

A. Fill in the Blanks

1. Let $z=\frac{\sin x/2+\cos x/2+i\tan x}{1+2i\sin x/2}$

$$=\frac{(\sin x/2+\cos x/2+i\tan x)(1-2i\sin x/2)}{(1+2i\sin x/2)(1-2i\sin x/2)}$$

$$=\frac{[\sin x/2+\cos x/2+2\sin x/2\tan x)+i(\tan x-2\sin^2 x/2-2\sin x/2\cos x/2)]}{(1+4\sin^2 x/2)}$$

But ATQ, $I_m(z)=0$ (as z is real)

$$\Rightarrow \tan x-2\sin\frac{x}{2}\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)=0$$

$$\Rightarrow \frac{\sin x}{\cos x}-2\sin^2 x/2-2\sin x/2\cos x/2=0$$

$$\Rightarrow \frac{\sin x}{\cos x}-(1-\cos x)-\sin x=0$$

$\Rightarrow \quad \sin x\left[\frac{1}{\cos x}-1\right]-[1-\cos x]=0$

$\Rightarrow \quad \left(\frac{1-\cos x}{\cos x}\right)\sin x-[1-\cos x]=0$

$\Rightarrow \quad (1-\cos x)\left(\frac{\sin x}{\cos x}-1\right)=0$

$\Rightarrow \quad \cos x=1 \quad \Rightarrow \quad x=2n\pi$ and

$\tan x=1 \quad \Rightarrow \quad x=n\pi+\pi/4$

$\therefore \quad x=2n\pi, n\pi+\pi/4$ **Ans.**

2. $|az_1-bz_2|^2+|bz_1+az_2|^2$

$= a^2|z_1|^2+b^2|z_2|^2-2ab\ \text{Re}(z_1\bar{z}_2)+b^2|z_1|^2$

$+a^2|z_2|^2+2ab\ \text{Re}(z_1\bar{z}_2)$

$=(a^2+b^2)(|z_1|^2+|z_2|^2)$

3. **KEY CONCEPT :** $|z_1-z_2|$ = distance between two points represented by z_1 and z_2.

As $z_1=a+i,\ z_2=1+bi$ and $z_3=0$ form an equilateral triangle, therefore $|z_1-z_3|=|z_2-z_3|=|z_1-z_2|$

$\Rightarrow \quad |a+i|=|1+bi|=|(a-1)+i(1-b)|$

$\Rightarrow \quad a^2+1=1+b^2=(a-1)^2+(1-b)^2$

$\Rightarrow \quad a^2=b^2=a^2+b^2-2a-2b+1$

$\Rightarrow \quad a=b$ ....(1)

($\because\ a,b>0 \quad \therefore \quad a\neq -b$) and

$b^2-2a-2b+1=0$

or $\quad a^2-2a-2b+1=0$ ....(2)

$\Rightarrow \quad a^2-2a-2a+1=0 \qquad [\because a=b]$

$\Rightarrow \quad a^2-4a+1=0$

$\Rightarrow \quad a=\frac{4\pm 2\sqrt{3}}{2}=2\pm\sqrt{3}$ But $0<a,b<1$

$\therefore \quad a=2-\sqrt{3}$ also $b=2-\sqrt{3}$

4. If we see the problem as in co-ordinate geometry we have $D\equiv(1,1)$ and $M\equiv(2,-1)$

We know that diagonals of rhombus bisect each other at 90°

$\therefore$ AC passes through M and is $\perp$ to BD

$\therefore$ Eq. of AC in symmetric form can be written as

$$\frac{x-2}{2/\sqrt{5}}=\frac{y+1}{1/\sqrt{5}}=r$$

Now for pt. A, as

$$AM=\frac{1}{2}DM=\frac{1}{2}\sqrt{(2-1)^2+(-1-1)^2}=\sqrt{5}/2$$

Putting $r=\pm\sqrt{5}/2$ *we get*,

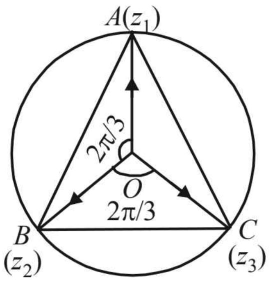
$$\frac{x-2}{2/\sqrt{5}}=\frac{y+1}{1/\sqrt{5}}=\pm\sqrt{5}/2$$

$\Rightarrow \quad x=\pm 1+2,\ y=\pm\frac{1}{2}-1$

$\Rightarrow \quad x=3$ or $1, y=\frac{-1}{2}$ *or* $\frac{-3}{2}$

$\therefore$ Pt. A is $3-i/2$ or $1-(3/2)i$

5. Let z_1, z_2, z_3 be the vertices A, B and C respectively of equilateral ΔABC, inscribed in a circle $|z|=2$, centre $(0,0)$ rasius $=2$



Given $\quad z_1=1+i\sqrt{3}$

$z_2=e^{\frac{2\pi i}{3}}z_1$

$=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)(1+i\sqrt{3})$

$=\frac{-1-3}{2}=-2$

and $z_3=e^{4(\pi/3)i}z_1$

$=\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)(1+i\sqrt{3})$

$=\left(\frac{-1-i\sqrt{3}}{2}\right)(1+i\sqrt{3})=\frac{-1-2i\sqrt{3}+3}{2}=1-i\sqrt{3}$

6. rth term of the given series,

$=r[(r+1)-\omega](r+1)-\omega^2]$

$=r[(r+1)^2-(\omega+\omega^2)(r+1)+\omega^3]$

$=r[(r+1)^2-(-1)(r+1)+1]$

$=r[(r^2+3r+3]=r^3+3r^2+3r$

Thus, sum of the given series,

$$=\sum_{r=1}^{(n-1)}(r^3+3r^2+3r)$$

$=\frac{1}{4}(n-1)^2n^2+3.\frac{1}{6}(n-1)(n)(2n-1)+3.\frac{1}{2}(n-1)n$

$=(n-1)(n)\left[\frac{1}{4}(n-1)n+\frac{1}{2}(2n-1)+\frac{3}{2}\right]$

$=\frac{1}{4}(n-1)\ n[n^2-n+4n-2+6]$

$=\frac{1}{4}(n-1)n[n^2+3n+4]$

B. True / False

1. Let $z = x + iy$

then $1 \cap z \Rightarrow 1 \le x \,\&\, 0 \le y$ (by def.)

Consider

$$\frac{1-z}{1+z} = \frac{1-(x+iy)}{1+(x+iy)} = \frac{(1-x)-iy}{(1+x)+iy} \times \frac{(1+x)-iy}{(1+x)-iy}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{iy(1-x+1+x)}{(1+x)^2+y^2}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2} - \frac{2iy}{(1+x)^2+y^2}$$

$$\frac{1-z}{1+z} \cap 0 \quad \Rightarrow \quad \frac{1-x^2-y^2}{(1+x)^2+y^2} \le 0$$

and $\frac{-2y}{(1+x)^2+y^2} \le 0$

$\Rightarrow \; 1 - x^2 - y^2 \le 0$ and $-2y \le 0$

$\Rightarrow \; x^2 + y^2 \ge 1$ and $y \ge 0$ which is true as

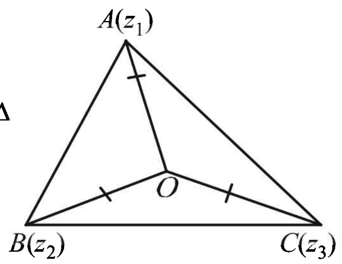
$x \ge 1 \;\&\; y \ge 0$

$\therefore$ The given statement is true $\forall \; z \in C$.

2. As $|z_1| = |z_2| = |z_3|$

$\therefore$ z_1, z_2, z_3 are equidistant from origin. Hence O is the circumcentre of ΔABC.

But according to question ΔABC is equilateral and we know that in an equilateral Δ circumcentre and centroid coincide.



$\therefore$ Centroid of $\Delta ABC = 0$

$$\Rightarrow \frac{z_1 + z_2 + z_3}{3} = 0 \Rightarrow z_1 + z_2 + z_3 = 0$$

$\therefore$ Statement is true.

3. If z_1, z_2, z_3 are in A.P. then, $\frac{z_1 + z_3}{2} = z_2$

$\Rightarrow$ z_2 is mid pt. of line joining z_1 and z_3.

$\Rightarrow$ z_1, z_2, z_3 lie on a st. line

$\therefore$ Given statement is false

4. $\because$ Cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}$

$\therefore$ Vertices of triangle are

$$A(1,0), B\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right), C\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$$

$\Rightarrow AB = BC = CA \quad \therefore \Delta$ is equilateral.

C. MCQs with ONE Correct Answer

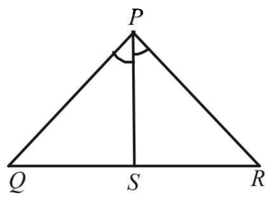
1. **(b)** $(x-1)^3 + 8 = 0$

$\Rightarrow (x-1)^3 = -8 = (-2)^3$

$\Rightarrow x - 1 = -2$

or -2ω or $-2\omega^2$

$\Rightarrow x = -1, 1-2\omega, 1-2\omega^2$



2. **(d)** $\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1-1+2i}{2} = i$

Now $i^n = 1 \Rightarrow$ the smallest positive integral value of n should be 4.

3. **(a)** ATQ $|x + iy - 5i| = |x + iy + 5i|$

$\Rightarrow |x + (y-5)i| = |x + (y+5)i|$

$\Rightarrow x^2 + (y-5)^2 = x^2 + (y+5)^2$

$\Rightarrow x^2 + y^2 - 10y + 25 = x^2 + y^2 + 10y + 25$

$\Rightarrow 20y = 0 \Rightarrow y = 0$

$\therefore$ 'a' is the correct alternative.

4. **(b)** $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = -i\left(\frac{-1}{2} + \frac{i\sqrt{3}}{2}\right) = i\omega$

$$\frac{\sqrt{3}}{2} - \frac{i}{2} = i\left(\frac{-1}{2} - \frac{i\sqrt{3}}{2}\right) = i\omega^2$$

$\therefore \quad z = (-i\omega)^5 + (i\omega^2)^5 = -i\omega^2 + i\omega$

$= i(\omega - \omega^2) = i(i\sqrt{3}) = -\sqrt{3}$

$\Rightarrow$ Re$(z) < 0$ and Im$(z) = 0$

$\therefore$ (b) is the correct choice.

5. **(d)** $|z - 4| < |z - 2|$

$\Rightarrow |(x-4) + iy| < |(x-2) + iy|$

$\Rightarrow (x-4)^2 + y^2 < (x-2)^2 + y^2$

$\Rightarrow -8x + 16 < -4x + 4 \Rightarrow 4x - 12 > 0$

$\Rightarrow x > 3 \Rightarrow$ Re$(z) > 3$

6. **(b)** $|\omega| = 1 \Rightarrow \left|\frac{1-iz}{z-i}\right| = 1$

$\Rightarrow |1 - iz| = |z - i|$

$\Rightarrow |1 - i(x+iy)| = |x + iy - i|$

$\Rightarrow |(y+1) - ix| = |x + i(y-1)|$

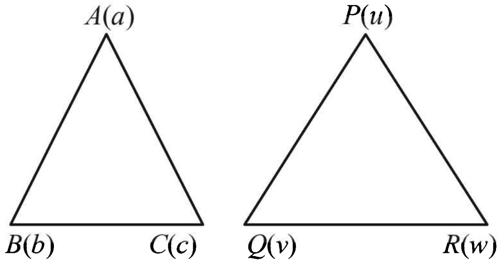
$\Rightarrow x^2 + (y+1)^2 = x^2 + (y-1)^2$

$\Rightarrow 4y = 0 \Rightarrow y = 0 \Rightarrow z$ lies on real axis

7. **(b)** If vertices of a parallelogram are z_1, z_2, z_3, z_4 then as diagonals bisect each other

$$\therefore \frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2} \Rightarrow z_1 + z_3 = z_2 + z_4$$

8. (b) Let ABC be the Δ with vertices a, b, c and PQR be the Δ with vertices u, v, w.
Then $c = (1-r)\,a + rb$



$\Rightarrow \quad c - a = r(b-a) \quad \Rightarrow \quad \dfrac{c-a}{b-a} = r \quad$ ....(1)

$\Rightarrow \quad w = (1-r)\,u + rv \quad \Rightarrow \quad \dfrac{w-u}{v-u} = r \quad$ ....(2)

From (1) and (2) $\left|\dfrac{c-a}{b-a}\right| = \left|\dfrac{w-u}{v-u}\right|$ and

$\arg\left(\dfrac{c-a}{b-a}\right) = \arg\left(\dfrac{w-u}{v-u}\right)$

$\Rightarrow \quad \dfrac{AC}{AB} = \dfrac{PR}{PQ}$ and $\angle CAB = \angle RPQ$

$\Rightarrow \quad \Delta ABC \sim \Delta PQR$

9. (b) $(1+\omega)^7 = A + B\omega$

$\Rightarrow \quad (-\omega^2)^7 = A + B\omega \qquad (\because\ 1+\omega+\omega^2 = 0)$

$\Rightarrow \quad -\omega^{14} = A + B\omega$

$\Rightarrow \quad -\omega^2 = A + B\omega \qquad (\because\ \omega^3 = 1)$

$\Rightarrow \quad 1+\omega = A + B\omega \Rightarrow A = 1, B = 1$

10. (d) $\because\ |z| = |\omega|$ and $\arg z = \pi - \arg \omega$

Let $\omega = re^{i\theta}$ then $z = re^{i(\pi-\theta)}$

$\Rightarrow \quad z = re^{i\pi} . e^{-i\theta}$

$= (re^{-i\theta})\,(\cos \pi + i \sin \pi) = \bar{\omega}\,(-1) = -\bar{\omega}$

11. (c) Given that $|z + i\omega| = |z - i\bar{\omega}|$

$\Rightarrow \quad |z - (-i\omega)| = |z - (-i\bar{\omega})|$

$\Rightarrow$ z lies on perpendicular bisector of the line segment joining $(-i\omega)$ and $(-i\bar{\omega})$, which is real axis, $(-i\omega)$ and $(-i\bar{\omega})$ being mirror images of each other.

$\therefore$ Im$(z) = 0$.

If $z = x$ then $|z| \le 1 \Rightarrow x^2 \le 1 \Rightarrow -1 \le x \le 1$

$\therefore$ (c) is the correct option.

12. (d) $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$

$= (1+i)^{n_1} + (1-i)^{n_1} + (1+i)^{n_2} + (1-i)^{n_2}$

Using $1 + i = \sqrt{2}\,(\cos \pi/4 + i \sin \pi/4)$

and $\quad 1 - i = \sqrt{2}\,(\cos \pi/4 - i \sin \pi/4)$

We get the given expression as

$$= (\sqrt{2})^{n_1}\left[\cos\frac{n_1\pi}{4} + i\sin\frac{n_1\pi}{4}\right]$$

$$+ (\sqrt{2})^{n_2}\left[\cos\frac{n_2\pi}{4} + i\sin\frac{n_2\pi}{4}\right]$$

$$+ (\sqrt{2})^{n_2}\left[\cos\frac{n_2\pi}{4} - i\sin\frac{n_2\pi}{4}\right]$$

$$= (\sqrt{2})^{n_1}\left[2\cos\frac{n_1\pi}{4}\right] + (\sqrt{2})^{n_2}\left[2\cos\frac{n_2\pi}{4}\right]$$

= real number irrespective the values of n_1 and n_2

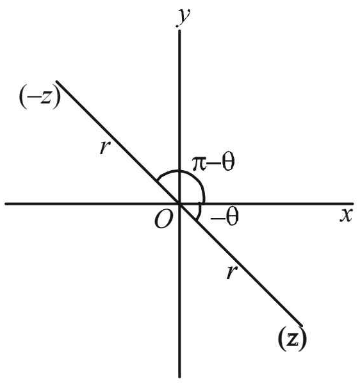
$\therefore$ (d) is the most appropriate answer.

13. (c) $E = 4 + 5(\omega)^{334} + 3(\omega)^{365} = 4 + 5\omega + 3\omega^2$

$= 1 + 2\omega + 3(1 + \omega + \omega^2) = 1 + (-1 + i\sqrt{3}) = i\sqrt{3}$

14. (a) $\arg(z) < 0$ (given) $\Rightarrow \arg(z) = -\,\theta$

Now



$z = r\cos(-\theta) + i\sin(-\theta) = r[\cos(\theta) - i\sin(\theta)]$

Again $-z = -r\,[\cos(\theta) - i\sin(\theta)]$

$= r\,[\cos(\pi-\theta) + i\sin(\pi-\theta)]$

$\therefore \quad \arg\ (-z) = \pi - \theta;$

Thus $\arg\ (-z) - \arg(z) = \pi - \theta - (-\theta) = \pi - \theta + 0 = \pi$

15. (a) $|z_1| = |z_2| = |z_3| = 1$ (given)

Now, $\quad |z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\bar{z}_1 = 1$

Similarly $z_2\bar{z}_2 = 1,\ z_3\bar{z}_3 = 1$

Now, $\left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1 \Rightarrow |\bar{z}_1 + \bar{z}_2 + \bar{z}_3| = 1$

$\Rightarrow \quad \left|\overline{z_1 + z_2 + z_3}\right| = 1$ **NOTE THIS STEP**

$\Rightarrow \quad |z_1 + z_2 + z_3| = 1$

16. (d) Let $z = (1)^{1/n} = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, k = 0, 1, 2, ..., n-1.$$

Let $\quad z_1 = \cos\left(\frac{2k_1\pi}{n}\right) + i\sin\left(\frac{2k_1\pi}{n}\right)$

and $\quad z_2 = \cos\left(\frac{2k_2\pi}{n}\right) + i\sin\frac{2k_2\pi}{n}$

be the two values of z. s.t. they subtend $\angle$ of 90° at origin.

$$\therefore \frac{2k_1\pi}{n} - \frac{2k_2\pi}{n} = \pm\frac{\pi}{2} \Rightarrow 4(k_1 - k_2) = \pm n$$

As k_1 and k_2 are integers and $k_1 \neq k_2$.

$\therefore \quad n = 4k, k \in I$

17. (c) $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$

$\Rightarrow \arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \arg\left(\frac{1 - i\sqrt{3}}{2}\right)$

$\Rightarrow \arg\ (\cos(-\pi/3) + i\sin(-\pi/3))$

$\Rightarrow$ angle between $z_1 - z_3$ and $z_2 - z_3$ is 60°.

and $\left|\frac{z_1 - z_3}{z_2 - z_3}\right| = \left|\frac{1 - i\sqrt{3}}{2}\right|$

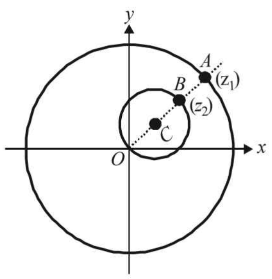
$\Rightarrow \left|\frac{z_1 - z_3}{z_2 - z_3}\right| = 1 \Rightarrow |z_1 - z_3| = |z_2 - z_3|$

NOTE THIS STEP

$\Rightarrow$ The Δ with vertices z_1, z_2 and z_3 is isosceles with vertical $\angle 60°$. Hence rest of the two angles should also be 60° each.

$\Rightarrow$ Req. Δ is an equilateral Δ.

18. (b) $|z_1| = 12 \Rightarrow z_1$ lies on a circle with centre (0, 0) and radius 12 units, and $|z_2 - 3 - 4i| = 5 \Rightarrow z_2$ lies on a circle with centre (3, 4) and radius 5 units.



From fig. it is clear that $|z_1 - z_2|$ i.e., distance between z_1 and z_2 will be min when they lie at A and B resp. i.e., O, C, B, A are collinear as shown. Then $z_1 - z_2 = AB = OA - OB = 12 - 2(5) = 2$. As above is the min, value, we must have $|z_1 - z_2| \geq 2$.

19. (a) Given that $|z| = 1$ and $\omega = \frac{z-1}{z+1} (z \neq -1)$

Now we know that $z\bar{z} = |z|^2$

$\Rightarrow \quad z\bar{z} = 1 \qquad$ (for $|z| = 1$)

$$\therefore \quad \omega = \left(\frac{z-1}{z+1}\right) \times \frac{(\bar{z}+1)}{(\bar{z}+1)} = \frac{z\bar{z} + z - \bar{z} - 1}{z\bar{z} + z + \bar{z} + 1} = \frac{2iy}{2 + 2x}$$

$[\because z\bar{z} = 1$ and taking $z = x + iy$ so that

$z + \bar{z} = 2x$ and $z - \bar{z} = 2iy]$

$\Rightarrow \quad \text{Re}(\omega) = 0$

20. (b) $(1 + \omega^2)^n = (1 + \omega^4)^n$

$$\Rightarrow (-\omega)^n = (1 + \omega)^n = (-\omega^2)^n \Rightarrow \omega^n = 1 \Rightarrow n = 3$$

21. (a) Here we observe that.

$AB = AC = AD = 2$

$\therefore$ BCD is an arc of a circle with centre at A and radius 2. Shaded region is outer (exterior) part of this sector $ABCDA$.

$\therefore$ For any pt. z on arc BCD we should have

$|z - (-1)| = 2$

and for shaded region, $|z + 1| > 2$ ....(i)

For shaded region we also have

$-\pi/4 < \arg(z + 1) < \pi/4$

or $\quad |\arg(z + 1)| < \pi/4$ ...(ii)

Combining (i) and (ii), (a) is the correct option.

22. (b) Given that a, b, c are integers not all equal, ω is cube root of unity $\neq$ 1, then

$$|a + b\omega + c\omega^2|$$

$$= \left|a + b\left(\frac{-1 + i\sqrt{3}}{2}\right) + c\left(\frac{-1 - i\sqrt{3}}{2}\right)\right|$$

$$= \left|\left(\frac{2a - b - c}{2}\right) + i\left(\frac{b\sqrt{3} - c\sqrt{3}}{2}\right)\right|$$

$$= \frac{1}{2}\sqrt{(2a - b - c)^2 + 3(b - c)^2}$$

$$= \sqrt{\frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2]}$$

R.H.S. will be min. when $a = b = c$, but we cannot take $a = b = c$ as per question.

$\therefore$ The min value is obtained when any two are zero and third is a minimum magnitude integer i.e. 1.
Thus $b = c = 0, a = 1$ gives us the minimum value 1.

23. (b) Operating $R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 3 & 0 & 0 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = 3[-\omega - 1 - \omega] = 3(\omega^2 - \omega)$$

$= 3\omega\,(\omega - 1)$

24. (d) $\because \dfrac{w - \bar{w}z}{1 - z}$ is purely real

$$\therefore \overline{\left(\frac{w-\bar{w}z}{1-z}\right)} = \left(\frac{w-\bar{w}z}{1-z}\right) \Rightarrow \frac{\bar{w}-w\bar{z}}{1-\bar{z}} = \frac{w-\bar{w}z}{1-z}$$

$\Rightarrow \bar{w} - \bar{w}z - w\bar{z} + wz\bar{z} = w - w\bar{z} - \bar{w}z + \bar{w}z\bar{z}$

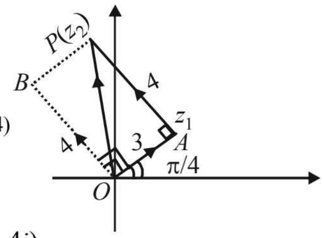
$\Rightarrow w - \bar{w} = (w - \bar{w})\,|z|^2$

$\Rightarrow |z|^2 = 1 \quad (\because w = \alpha + i\beta \text{ and } \beta \neq 0)$

$\Rightarrow |z| = 1$ also given $z \neq 1$

$\therefore$ The required set is $\{z : |z| = 1, z \neq 1\}$

25. (d) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$



$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$

$\Rightarrow \overrightarrow{OP} = 3e^{i\pi/4} + 4e^{i(\pi/2 + \pi/4)}$

$= 3e^{i\pi/4} + 4e^{i\pi/2}.e^{i\pi/4}$

$= 3e^{i\pi/4} + 4ie^{i\pi/4} = e^{i\pi/4}(3 + 4i).$

26. (d) Given $|z| = 1$ and $z \neq \pm 1$

To find locus of $\omega = \dfrac{z}{1 - z^2}$

We have $\omega = \dfrac{z}{1 - z^2} = \dfrac{z}{z\bar{z} - z^2}$

$[\because |z| = 1 \Rightarrow |z|^2 = z\bar{z} = 1]$

$= \dfrac{1}{\bar{z} - z} =$ purely imaginary number

$\therefore$ ω must lie on y-axis.

27. (d) The initial position of point is $Z_0 = 1 + 2i$

$\therefore Z_1 = (1 + 5) + (2 + 3)i = 6 + 5i$

Now Z_1 is moved through a distance of $\sqrt{2}$ units in the direction $\hat{i} + \hat{j}$. (i.e. by $1 + i$)

$\therefore$ It becomes $Z_1' = Z_1 + (1 + i) = 7 + 6i$

Now OZ_1' is rotated through an angle $\dfrac{\pi}{2}$ in anticlockwise direction, therefore $Z_2 = iZ_1' = -6 + 7i$

28. (d) $z = \cos\theta + i\sin\theta$

$\Rightarrow z^{2m-1} = (\cos\theta + i\sin\theta)^{2m-1}$

$= \cos(2m-1)\theta + i\sin(2m-1)\theta$

$\left[\begin{array}{l}\text{using De Moivre's theorem} \\ (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta\end{array}\right]$

$\therefore \operatorname{Im}(z^{2m-1}) = \sin(2m-1)\theta$

$$\therefore \sum_{m=1}^{15} \operatorname{Im}(z^{2m-1}) = \sum_{m=1}^{15} \sin(2m-1)\theta$$

$= \sin\theta + \sin 3\theta + \sin 5\theta + \ldots\ldots + \text{upto } 15 \text{ terms}$

$$= \frac{\sin\left[15\left(\frac{2\theta}{2}\right)\right].\sin[\theta + 14\times\theta]}{\sin\theta}$$

$$\left[\begin{array}{l}\text{Using } \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots .n \text{ terms} \\ = \dfrac{\sin(n\beta/2).\sin[\alpha + (n-1)\beta/2]}{\sin(\beta/2)}\end{array}\right]$$

$$= \frac{\sin 15\theta.\sin 15\theta}{\sin\theta} = \frac{\sin 30°.\sin 30°}{\sin 2°} = \frac{1}{4\sin 2°}$$

29. (a) Given $z = x + iy$ where x and y are integer

Also $z\bar{z}^3 + \bar{z}z^3 = 350 \Rightarrow |z|^2\,(\bar{z}^2 + z^2) = 350$

$\Rightarrow (x^2 + y^2)(x^2 - y^2) = 175$

$\Rightarrow (x^2 + y^2)(x^2 - y^2) = 25 \times 7$ ...(i)

or $(x^2 + y^2)(x^2 - y^2) = 35 \times 5$ ...(ii)

$\because$ x and y are integers,

$\therefore x^2 + y^2 = 25$ and $x^2 - y^2 = 7$ [From eq (i)]

$\Rightarrow x^2 = 16$ and $y^2 = 9$

$\Rightarrow x = \pm 4$ and $y = \pm 3$

$\therefore$ Vertices of rectangle are $(4, 3), (4, -3), (-4, -3), (-4, 3)$.

So, area of rectangle $= 8 \times 6 = 48$ sq. units

Now from eq. (ii)

or $x^2 + y^2 = 35$ and $x^2 - y^2 = 5$

$\Rightarrow x^2 = 20$, which is not possible for any integral value of x

30. (d) $\because$ $Im(z) \neq 0 \Rightarrow z$ is non real

and equation $z^2 + z + (1 - a) = 0$

will have non real roots, if $D < 0$

$\Rightarrow 1 - 4(1 - a) < 0 \Rightarrow 4a < 3 \Rightarrow a < \dfrac{3}{4}$

$\therefore$ a can not take the value $\dfrac{3}{4}$

31. **(c)** As α lies on the circle $(x-x_0)^2+(y-y_0)^2=r^2$

$\therefore\ |\alpha-z_0|^2=r^2$

$\Rightarrow\ (\alpha-z_0)(\bar{\alpha}-\bar{z}_0)=r^2$

$\Rightarrow \alpha\bar{\alpha}-\alpha\bar{z}_0-\bar{\alpha}z_0+z_0\bar{z}_0=r^2$

$\Rightarrow |\alpha|^2+|z_0|^2-\alpha\bar{z}_0-\bar{\alpha}z_0=r^2$ (i)

Also $\frac{1}{\bar{\alpha}}$ lies on the circle $(x-x_0)^2+(y-y_0)^2=4r^2$

$\therefore\ \left|\frac{1}{\bar{\alpha}}-z_0\right|^2=4r^2 \Rightarrow \left(\frac{1}{\bar{\alpha}}-z_0\right)\left(\frac{1}{\alpha}-\bar{z}_0\right)=4r^2$

$\Rightarrow \frac{1}{\alpha\bar{\alpha}}-\frac{z_0}{\alpha}-\frac{\bar{z}_0}{\bar{\alpha}}+z_0\bar{z}_0=4r^2$

$\Rightarrow \frac{1}{|\alpha|^2}-\frac{z_0\bar{\alpha}}{|\alpha|^2}-\frac{\bar{z}_0\alpha}{|\alpha|^2}+|z_0|^2=4r^2$

$\Rightarrow 1+|\alpha|^2|z_0|^2-z_0\bar{\alpha}-\bar{z}_0\alpha=4r^2|\alpha|^2$ (ii)

Subtracting eq^n (i) from (ii) we get

$1-|\alpha|^2+|z_0|^2(|\alpha|^2-1)=r^2(4|\alpha|^2-1)$

or $(|\alpha|^2-1)(|z_0|^2-1)=r^2(4|\alpha|^2-1)$

Using $|z_0|^2=\frac{r^2+2}{2}$ we get

$(|\alpha|^2-1)\frac{r^2}{2}=r^2(4|\alpha|^2-1)$

$\Rightarrow |\alpha|^2-1=8|\alpha|^2-2 \Rightarrow |\alpha|=\frac{1}{\sqrt{7}}$

D. MCQs with ONE or MORE THAN ONE Correct

1. **(a, b, c)** $z_1=a+ib$ and $z_2=c+id$.

ATQ $|zi|^2=|z_2|^2=1$

$\Rightarrow a^2+b^2=1$ and $c^2+d^2=1$. ....(1)

Also Re $(z_1\,\overline{z_2})=0 \Rightarrow ac+bd=0$

$\Rightarrow \frac{a}{b}=\frac{-d}{c}=\alpha\ (say)$ ....(2)

From (1) and (2), we get

$b^2\alpha^2+b^2=c^2\alpha^2+c^2 \Rightarrow b^2=c^2;$

Similarly $a^2=d^2$

$\therefore\ |\omega_1|=\sqrt{a^2+c^2}=\sqrt{c^2+d^2}=1$

and $|\omega_2|=\sqrt{b^2+d^2}=\sqrt{c^2+d^2}=1$

Also Re $(\omega_1\bar{\omega}_2)=ab+cd=(b\alpha)b+c(-c\alpha)$

$=\alpha(b^2-c^2)=0$

2. **(a, d)** Let $z_1=a+ib,\ a>0$ and $b\in R;\ z_2=c+id,$ $d<0, c\in R$

then $|z_1|=|z_2| \Rightarrow a_2+b_2=c_2+d_2$

$\Rightarrow a^2-c^2=d^2-b^2$ ....(1)

Now, $\frac{z_1+z_2}{z_1-z_2}=\frac{(a+c)+i(b+d)}{(a-c)+i(b-d)}$

$=\frac{[(a^2-c^2)+(b^2-d^2)]+i[(a-c)(b+d)-(a+c)(b-d)]}{(a-c)^2+(b-d)^2}$

$=\frac{i[(a-c)(b+d)-(a+c)(b-d)]}{(a-c)^2+(b-d)^2}$ [Using (1)]

= purely imaginary number or zero in case $a+c=b+d=0.$

3. **(c)** Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$

and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$

where $r_1=|z_1|,\ r_2=|z_2|,\ \theta_1=\arg(z_1),\ \theta_2=\arg(z_2)$

$\therefore\ z_1+z_2=r_1(\cos\theta_1+i\sin\theta_1)+r_2(\cos\theta_2+i\sin\theta_2)$

$=(r_1\cos\theta_1+r_2\cos\theta_2)+i(r_1\sin\theta_1+r_2\sin\theta_2)$

$=r_1^2\cos^2\theta_1+r_2^2\cos^2\theta_2+2r_1r_2\cos\theta_1\cos\theta_2$
$+r_1^2\sin^2\theta_1+r_2^2\sin^2\theta_2+2r_1r_2\sin\theta_1\sin\theta_2$

$=r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)$

and $|z_1|+|z_2|=r_1+r_2$

Since $|z_1+z_2|=|z_1|+|z_2|$ (given)

$\Rightarrow |z_1+z_2|^2=|z_1|^2+|z_2|^2+2|z_1||z_2|$

$\Rightarrow r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)=r_1^2+r_2^2+2r_1r_2$

$\Rightarrow \cos(\theta_1-\theta_2)=1 \Rightarrow \theta_1-\theta_2=0$

$\Rightarrow \operatorname{Arg}(z_1)=\arg(z_2)$

4. **(d)** Let $z=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$

Then by DeMoivre's theorem, we have

$z^k=\cos\frac{2\pi k}{7}+i\sin\frac{2\pi k}{7}$

Now, $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7}\right)$

$=\sum_{k=1}^{6}(-i)\left(\cos\frac{2\pi k}{7}+i\sin\frac{2\pi k}{7}\right)$

$=(-i)\sum_{k=1}^{6}z^k=-i\,z\frac{(1-z^6)}{1-z}=-i\left(\frac{z-z^7}{1-z}\right)$

$= (-i)\left(\frac{z-1}{1-z}\right) = $ [Using $z^7 = \cos 2\pi + i \sin 2\pi = 1$]

$= i\left(\frac{1-z}{1-z}\right) = i$

5. (d) We have $(1+\omega+\omega^2)^7 = (-\omega^2 - \omega^2)^7$

$(-2)^7 (\omega^2)^7 = -128\omega^{14} = -128\omega^2$

6. (b) $\sum_{i=1}^{13}(i^n + i^{n+1}) = \sum_{i=1}^{13} i^n (1+i) = (1+i)\sum_{i=1}^{13} i^n$

This forms a G.P.

Sum of G.P. $= i\,(1+i)\,\frac{(1-i^{13})}{1-i} = i - 1$ as $i^{13} = i$

7. (d) Taking $-3i$ common from C_2, we get

$$-3i\begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = 0 \quad (\because C_2 \equiv C_3)$$

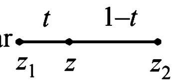
$\Rightarrow x = 0,\ y = 0$

8. (a,c,d) Given that $z = (1-t)\,z_1 + t\,z_2$ where $0 < t < 1$

$\Rightarrow z = \frac{(1-t)z_1 + tz_2}{(1-t)+t}$

$\Rightarrow$ z divides the join of z_1 and z_2 internally in the ratio $t : (1-t)$.

$\therefore z_1, z$ and z_2 are collinear



$\Rightarrow |z - z_1| + |z - z_2| = |z_1 - z_2|$

Also $z = (1-t)z_1 + t\,z_2$

$\Rightarrow \frac{z - z_1}{z_2 - z_1} = t =$ purely real number

$\therefore \arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \Rightarrow \arg(z - z_1) = \arg(z_2 - z_1)$

Also $\frac{z - z_1}{z_2 - z_1} = t \Rightarrow \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} = t$

$\Rightarrow \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$

$\Rightarrow (z - z_1)(\bar{z}_2 - \bar{z}_1) = (\bar{z} - \bar{z}_1)(z_2 - z_1)$

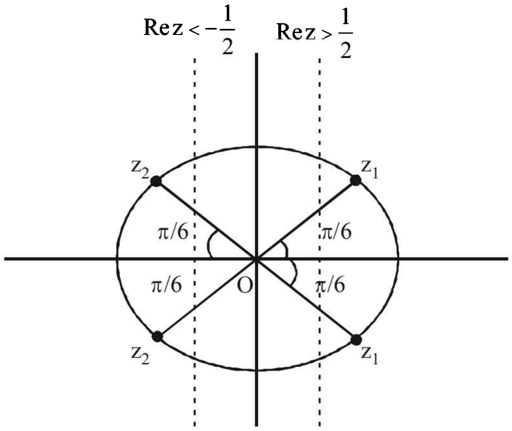
$$\Rightarrow \begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$$

9. (c, d) $w = \frac{\sqrt{3}+i}{2} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

and $w^n = \cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}$

$\therefore$ P contains all those points which lie on unit circle and have arguments $\frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}$ and so on.

As $z_1 \in P \cap H_1$ and $z_2 \in P \cap H_2$, therefore z_1 and z_2 can have possible positions as shown in the figure.



$\therefore \angle z_1Oz_2$ can be $\frac{2\pi}{3}$ or $\frac{5\pi}{6}$.

10. (a, c, d) $z = \frac{1}{a+ibt} = x + iy$

$\Rightarrow x + iy = \frac{a - ibt}{a^2 + b^2t^2}$

$\Rightarrow x = \frac{a}{a^2 + b^2t^2},\ y = \frac{-bt}{a^2 + b^2t^2}$

$\Rightarrow x^2 + y^2 = \frac{1}{a^2 + b^2t^2} = \frac{x}{a}$

$\Rightarrow x^2 + y^2 - \frac{x}{a} = 0$

$\therefore$ Locus of z is a circle with centre $\left(\frac{1}{2a}, 0\right)$ and radius $= \frac{1}{2|a|}$ irrespective of 'a' +ve or –ve

Also for $b = 0, a \neq 0$, we get, $y = 0$

$\therefore$ locus is x-axis

and for $a = 0, b \neq 0$ we get $x = 0$

$\Rightarrow$ locus is y-axis.

$\therefore$ a, c, d are the correct options.

E. Subjective Problems

1. $\dfrac{1}{1-\cos\theta+2i\sin\theta}$

$$=\frac{1}{2\sin^2\theta/2+4i\sin\theta/2\cos\theta/2}=\frac{1}{2\sin\theta/2}$$

$$\left[\frac{\sin\theta/2-2i\cos\theta/2}{(\sin\theta/2+2i\cos\theta/2)(\sin\theta/2-2i\cos\theta/2)}\right]$$

$$=\frac{1}{2\sin\theta/2}\left[\frac{\sin\theta/2-2i\cos\theta/2}{(\sin^2\theta/2+4\cos^2\theta/2}\right]$$

$$=\frac{1}{2\sin\theta/2}\left[\frac{2\sin\theta/2-4i\cos\theta/2}{1-\cos\theta+4+4\cos\theta}\right]$$

$$=\frac{2}{2\sin\theta/2}\left[\frac{2\sin\theta/2-2i\cos\theta/2}{5+3\cos\theta}\right]$$

$$=\left(\frac{1}{5+3\cos\theta}\right)+\left(\frac{-2\cot\theta/2}{5+3\cos\theta}\right)i$$

which is of the form $X+iY$.

2. As β and γ are the complex cube roots of unity therefore, let $\beta=\omega$ and $\gamma=\omega^2$ so that $\omega+\omega^2+1=0$ and $\omega^3=1$.
Then $xyz=(a+b)(a\omega^2+b\omega)(a\omega+b\omega^2)$
$=(a+b)(a^2\omega^3+ab\omega^4+ab\omega^2+b^2\omega^3)$
$=(a+b)(a^2+ab\omega+ab\omega^2+b^2)$ (using $\omega^3=1$)
$=(a+b)(a^2+ab(\omega+\omega^2)+b^2)$
$=(a+b)(a^2-ab+b^2)$ (using $\omega+\omega^2=-1$)
$=a^3+b^3$ Hence proved.

3. Given $x+iy=\sqrt{\dfrac{c+ib}{c+id}}$

$$\Rightarrow (x+iy)^2=\frac{a+ib}{c+id} \quad ....(1)$$

Taking conjugate on both sides, we get

$$(x-iy)^2=\frac{a-ib}{c-id} \quad ....(2)$$

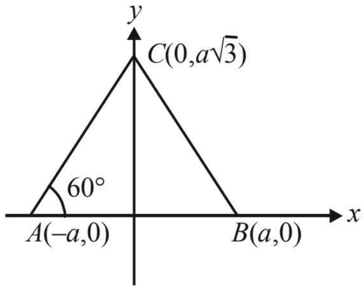
Multiply (1) and (2), we get

$$(x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$$

4. $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$

$\Rightarrow (4+2i)x-6i-2+(9-7i)y+3i-1=10i$
$\Rightarrow (4x+9y-3)+(2x-7y-3)i=10i$
$\Rightarrow 4x+9y-3=0$ and $2x-7y-3=10$
On solving these two, we get $x=3, y=-1$

5.



Let us consider the equilateral Δ with each side of length $2a$ and having two of its vertices on x-axis namely $A(-a,0)$ and $B(a,0)$, then third vertex C will clearly lie on y-axis s.t.
$OC=2a\sin 60°=a\sqrt{3}$ $\therefore$ C has the co-ordinates $(0,a\sqrt{3})$.
Now in the form of complex numbers if A, B and C are represented by z_1,z_2,z_3 then $z_1=-a$; $z_2=a$; $z_3=a\sqrt{3}\,i$
As in an equilateral Δ, centroid and circumcentre coincide, we get

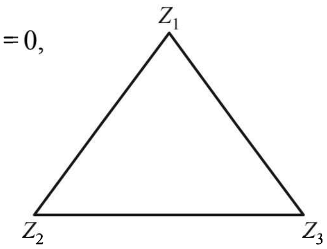
Circumcentre, $z_0=\dfrac{z_1+z_2+z_3}{3}$

$$\Rightarrow z_0=\frac{-a+a+a\sqrt{3}\,i}{3}=\frac{ia}{\sqrt{3}}$$

Now, $z_1^2+z_2^2+z_3^2=a^2+a^2-3a^2=-a^2$

and $3z_0^2=(ia)^2=-a^2$ $\therefore$ Clearly $3z_0^2=z_1^2+z_2^2+z_3^2$

6. We know that if z_1,z_2,z_3 are vertices of an equilateral Δ then



$\dfrac{z_1-z_2}{z_3-z_2}=\dfrac{z_3-z_1}{z_2-z_1}$ Here $z_3=0$,

We get $\dfrac{z_1-z_2}{-z_2}=\dfrac{-z_1}{z_2-z_1}$

$\Rightarrow -(z_1-z_2)^2=z_1z_2$

$\Rightarrow -z_1^2-z_2^2+2z_1z_2=z_1z_2 \Rightarrow z_1^2+z_2^2-z_1z_2=0.$

7. $1, a_1, a_2, ... a_{n-1}$ are the n roots of unity. Clearly above n values are roots of eq. $x^n-1=0$
Therefore we must have (by factor theorem)
$x^n-1=(x-1)(x-a_1)(x-a_2)....(x-a_{n-1})$ ....(1)

$$\Rightarrow \frac{x^n-1}{x-1}=(x-a_1)(x-a_2)....(x-a_{n-1}) \quad ....(2)$$

Differentiating both sides of eq. (1), we get
$nx^{n-1}=(x-a_1)(x-a_2)\ldots(x-a_{n-1})+(x-1)(x-a_2)$
$....(x-a_{n-1})+....+(x-1)(x-a_1)\ldots(x-a_{n-2})$
For $x=1$, we get $n=(1-a_1)(1-a_2)....(1-a_{n-1})$
[All the terms except first contain $(x-1)$ and hence become zero for $x=1$] Proved.

8. Let $A = z = x + iy$, $B = iz = -y + ix$,
$C = z + iz = (x - y) + i(x + y)$

Now, area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y & x & 1 \\ x-y & x+y & 1 \end{vmatrix}$

Operating $R_2 - R_1, R_3 - R_1$, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ -y-x & x-y & 0 \\ -y & x & 0 \end{vmatrix}$$

$$= \frac{1}{2}|x(-y-x) + y(x-y)|$$

$$= \frac{1}{2}|-xy - x^2 + xy - y^2| = \frac{1}{2}|-x^2 - y^2|$$

$$= \frac{1}{2}|x^2 + y^2| = \frac{1}{2}|z|^2$$ Hence Proved.

9. We are given that $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$

Also $\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$

$\Rightarrow \arg(z - z_1) - \arg(z - z_2) = \frac{\pi}{4}$ **NOTE THIS STEP**

$\Rightarrow \arg((x + iy) - (10 + 6i)) - \arg((x + iy) - (4 + 6i)) = \frac{\pi}{4}$

$\Rightarrow \arg[(x - 10) + i(y - 6)] - \arg[(x - 4) + i(y - 6)] = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{y-6}{x-10}\right) - \tan^{-1}\left(\frac{y-6}{x-4}\right) = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{\frac{y-6}{x-10} - \frac{y-6}{x-4}}{1 + \frac{(y-6)^2}{(x-4)(x-10)}}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{(x-4)(y-6) - (x-10)(y-6)}{(x-4)(x-10) + (y-6)^2} = \tan\frac{\pi}{4}$

$\Rightarrow (x - 4 - x + 10)(y - 6) = (x - 4)(x - 10) + (y - 6)^2$

$\Rightarrow 6y - 36 = x^2 + y^2 - 14x - 12y + 40 + 36$

$\Rightarrow x^2 + y^2 - 14x - 18y + 112 = 0$

$\Rightarrow (x^2 - 14x + 49) + (y^2 - 18y + 81) = 18$

$\Rightarrow (x - 7)^2 + (y - 9)^2 = (3\sqrt{2})^2$

$\Rightarrow |(x + iy) - (7 + 9i)| = 3\sqrt{2}$

$\Rightarrow |z - (7 + 9i)| = 3\sqrt{2}$. Hence Proved.

10. Dividing through out by i and knowing that $\frac{1}{i} = -i$, we get

$z^3 - iz^2 + iz + 1 = 0$

or $z^2(z - i) + i(z - i) = 0$ as $1 = -i^2$

or $(z - i)(z^2 + i) = 0$ $\therefore z = i$ or $z^2 = -i$

$\therefore$ $|z| = |i| = 1$ or $|z^2| = |z|^2 = |-i| = 1 \Rightarrow |z| = 1$

Hence in either case $|z| = 1$

11. Let $Z = r_1(\cos\theta_1 + i\sin\theta_1)$

and $W = r_2(\cos\theta_2 + i\sin\theta_2)$

We have $|Z| = r_1$, $|W| = r_2$, Arg $Z = \theta_1$ and

Arg $W = \theta_2$

Since $|Z| \le 1, |W| \le 1$, it follows that $r_1 \le$ *and* $r_2 \le 1$

We have $Z - W = (r_1\cos\theta_1 - r_2\cos\theta_2)$

$+ i(r_1\sin\theta_1 - r_2\sin\theta_2)$

$|Z - W|^2 = (r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2$

$= r_1^2\cos^2\theta_1 + r_2^2\cos^2\theta_2 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_1^2\sin^2\theta_1$

$+ r_2^2\sin^2\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2$

$= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2)$

$- 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$

$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$= (r_1 - r_2)^2 + 2r_1r_2[1 - \cos(\theta_1 - \theta_2)]$

$= (r_1 - r_2)^2 + 4r_1r_2\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$

$= |r_1 - r_2|^2 + 4r_1r_2\left|\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right|^2$

$\le |r_1 - r_2|^2 + 4\left|\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right|$ $[\because r_1, r_2 \le 1]$

But $|\sin\theta| \le |\theta| \ \forall\ \theta \in R$ **NOTE THIS STEP**

Therefore,

$|Z - W|^2 \le |r_1 - r_2|^2 + 4\left|\frac{\theta_1 - \theta_2}{2}\right|^2 \le |r_1 - r_2|^2 + |\theta_1 - \theta_2|^2$

Thus $|Z - W|^2 \le (|Z| - |W|)^2 + (\text{Arg } Z - \text{Arg } W)^2$

12. Let $z = x + iy$ then $\bar{z} = iz^2$

$\Rightarrow x - iy = i(x^2 - y^2 + 2ixy)$

$\Rightarrow x - iy = i(x^2 - y^2) - 2xy$

$\Rightarrow\ x(1+2y)=0\ ;\ x^2-y^2+y=0$

$\Rightarrow\ x=0$ or $y=-\dfrac{1}{2}\ \Rightarrow x=0,\ y=0,\ 1$

or $\quad y=-\dfrac{1}{2},\ x=\pm\dfrac{\sqrt{3}}{2}$

For non zero complex number z

$$x=0, y=1;\ x=\frac{\sqrt{3}}{2},\ y=-\frac{1}{2};\ x=\frac{-\sqrt{3}}{2},\ y=-\frac{1}{2}$$

$$\therefore\quad z=i,\ \frac{\sqrt{3}}{2}-\frac{i}{2},\ -\frac{\sqrt{3}}{2}-\frac{i}{2}$$

13. $z^2+pz+q=0$

$z_1+z_2=-p,\ z_1z_2=q$

By rotation through α in anticlockwise direction

$z_2=z_1e^{i\alpha}$ ....(1)

$$\frac{z_2}{z_1}=\frac{e^{i\alpha}}{1}=\frac{\cos\alpha+i\sin\alpha}{1}$$

Add 1 in both sides to get $z_1+z_2=-p$

$$\therefore\ \frac{z_1+z_2}{z_1}=\frac{1+\cos\alpha+i\sin\alpha}{1}=2\cos\frac{\alpha}{2}\left[\cos\frac{\alpha}{2}+i\sin\frac{\alpha}{2}\right]$$

or $\quad \dfrac{(z_2+z_1)}{z_1}=2\cos\dfrac{\alpha}{2}e^{i\alpha/2}$

On squaring $(z_2+z_1)^2=4\cos^2(\alpha/2)z_1^2.e^{i\alpha}$

$$=4\cos^2\frac{\alpha}{2}z_1^2.\frac{z_2}{z_1}=4\cos^2\frac{\alpha}{2}z_1z_2$$

or $\quad p^2=4q\cos^2\dfrac{\alpha}{2}$

14. Given that z and w are two complex numbers.

To prove $|z|^2w-|w|^2z=z-w \Leftrightarrow z=w$ or $z\bar{w}=1$

First let us consider

$|z|^2w-|w|^2z=z-w$ ....(1)

$\Rightarrow\ z(1+|w|^2=w(1+|z|^2)$

$\Rightarrow\ \dfrac{z}{w}=\dfrac{1+|z|^2}{1+|w|^2}=$ a real number

$\Rightarrow\ \overline{\left(\dfrac{z}{w}\right)}=\dfrac{z}{w}\Rightarrow\dfrac{\bar{z}}{\bar{w}}=\dfrac{z}{w}$

$\Rightarrow\ \bar{z}w=z\bar{w}$ ....(2)

Again from equation (1),

$z\bar{z}w-w\bar{w}z=z-w$

$z(\bar{z}w-1)-w(\bar{w}z-1)=0$

$z(z\bar{w}-1)-w(z\bar{w}-1)=0$ (Using equation (2))

$\Rightarrow\ (z\bar{w}-1)(z-w)=0\ \Rightarrow\ z\bar{w}=1$ or $z=w$

Conversely if $z=w$ then

L.H.S. of (1) $=|w|^2w-|w|^2w=0$

R.H.S. of (1) $=w-w=0$

$\therefore$ (1) holds

Also if $z\bar{w}=1$ then

L.H.S. of (1) $=z\bar{z}w-w\bar{w}z$

$=zz\bar{w}-w\bar{w}z=z-w=$ R.H.S. Hence proved.

15. The given equation can be written as

$(z^p-1)(z^q-1)=0$

$\therefore\quad z=(1)^{1/p}$ or $(1)^{1/q}$ ....(1)

where p and q are distinct prime numbers.

Hence both the equations will have distinct roots and as $z\neq1$, both will not be simultaneously zero for any value of z given by equations in (1)

NOTE THIS STEP

Also $1+\alpha+\alpha^2+...+\alpha^{p-1}=\dfrac{1-\alpha^p}{1-\alpha}=0\ (\alpha\neq1)$

or $\quad 1+\alpha+\alpha^2+...+\alpha^p=\dfrac{1-\alpha^q}{1-\alpha}=0\ (\alpha\neq1)$

Because of (1) either $\alpha^p=1$ and if $\alpha^q=1$ but not both simultaneously as p and q are distinct primes.

16. Given that $|z_1|<1<|z_2|$

Then $\left|\dfrac{1-z_1\bar{z}_2}{z_1-z_2}\right|<1$ is true

if $|1-z_1\bar{z}_2|<|z_1-z_2|$ is true

if $|1-z_1\bar{z}_2|^2<|z_1-z_2|^2$ is true

if $(1-z_1\bar{z}_2)\overline{(1-z_1\bar{z}_2)}<(z_1-z_2)\overline{(z_1-z_2)}$ is true

if $(1-z_1\bar{z}_2)(1-\bar{z}_1z_2)<(z_1-z_2)(\bar{z}_1-\bar{z}_2)$

if $1-z_1\bar{z}_2-\bar{z}_1z_2+z_1\bar{z}_1z_2\bar{z}_2<z_1\bar{z}_1-z_1\bar{z}_2$

$\quad -\bar{z}_1z_2+z_2\bar{z}_2$ is true

if $1+|z_1|^2|z_2|^2<|z_1|^2+|z_2|^2$ is true

if $(1-|z_1|^2)(1-|z_2|^2)<0$ is true.

which is obviously true

as $|z_1|<1<|z_2|$

$\Rightarrow\ |z_1|^2<1<|z_2|^2$

$\Rightarrow\ |1-|z_1|^2>0$ and $(1-|z_2|^2)<0$ Hence proved.

17. Let us consider, $\sum_{r=1}^{n}a_rz^r=1$ where $|a_r|<2$

$\Rightarrow\ a_1z+a_2z^2+a_3z^3+...+a_nz^n=1$

$\Rightarrow\ |a_1z+a_2z^2+a_3z^3+...+a_nz^n|=1$ ....(1)

But we know that $|z_1+z_2|\leq|z_1|+|z_2|$

$\therefore$ Using its generalised form, we get

$|a_1z+a_2z^2+a_3z^3+...+a_nz^n|$

$\leq|a_1z|+|a_2z^2|+...+|a_nz^n|$

$\Rightarrow \quad 1 \le |a_1||z| + |a_2|\,|z^2| + |a_3||z^3| + ... + |a_n||z^n|$

(Using eq^n(1))

But given that $|a_r| < 2 \,\forall\, r = 1(1)^n$

$\therefore \quad 1 < 2\,[\,|z| + |z|^2 + |z|^3 + ... + |z|^n]$

[Using $|z^n| = |z|^n$]

$\Rightarrow \quad 1 < 2\left[\dfrac{|z|(1-|z|^n)}{1-|z|}\right] \Rightarrow 2\left[\dfrac{|z|-|z|^{n+1}}{1-|z|}\right] > 1$

$\Rightarrow \quad 2\,[\,|z| - |z|^{n+1}] > 1 - |z| \quad (\because 1-|z| > 0 \text{ as } |z| < 1/3)$

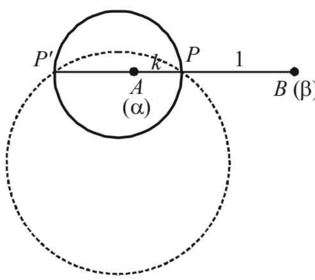
$\Rightarrow \quad [\,|z| - |z|^{n+1}] > \dfrac{1}{2} - \dfrac{1}{2}|z| \Rightarrow \dfrac{3}{2}|z| > \dfrac{1}{2} + |z|^{n+1}$

$\Rightarrow \quad |z| > \dfrac{1}{3} + \dfrac{2}{3}|z|^{n+1} \Rightarrow \quad |z| > \dfrac{1}{3}$

which is a contradiction as given that $|z| < \dfrac{1}{3}$

$\therefore$ There exist no such complex number.

18. We are given that



$$\left|\frac{z-\alpha}{z-\beta}\right| = k \Rightarrow |z-\alpha| = k\,|z-\beta|$$

Let pt. A represents complex number α and B that of β, and P represents z. then $|z-\alpha| = k\,|z-\beta|$

$\Rightarrow$ z is the complex number whose distance from A is k times its distance from B.

i.e. $PA = k\ PB$

$\Rightarrow$ P divides AB in the ratio $k : 1$ internally or externally (at P').

Then $\quad P\left(\dfrac{k\beta+\alpha}{k+1}\right)$ and $P'\left(\dfrac{k\beta-\alpha}{k-1}\right)$

Now through PP' there can pass a number of circles, but with given data we can find radius and centre of that circle for which PP' is diameter.

And hence then centre = mid. point of PP'

$$= \left(\frac{\frac{k\beta+\alpha}{k+1} + \frac{k\beta-\alpha}{k-1}}{2}\right) = \frac{k^2\beta + k\alpha - k\beta - \alpha + k^2\beta - k\alpha + k\beta - \alpha}{2(k^2-1)}$$

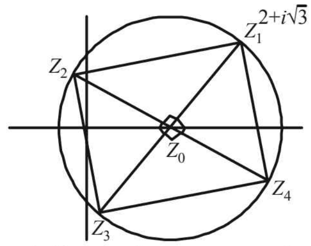
$$= \frac{k^2\beta - \alpha}{k^2-1} = \frac{\alpha - k^2\beta}{1-k^2}$$

Also radius

$$= \frac{1}{2}|PP'| = \frac{1}{2}\left|\frac{k\beta+\alpha}{k+1} - \frac{k\beta-\alpha}{k-1}\right|$$

$$= \frac{1}{2}\left|\frac{k^2\beta + k\alpha - k\beta - \alpha - k^2\beta + k\alpha - k\beta + \alpha}{k^2-1}\right| = \frac{k\,|\alpha-\beta|}{|1-k^2|}$$

19. The given circle is $|z-1| = \sqrt{2}$ where z_0=1 is the centre and $\sqrt{2}$ is radius of circle. z_1 is one of the vertex of square inscribed in the given circle.



Clearly z_2 can be obtained by rotating z_1 by an $\angle 90°$ in anticlockwise sense, about centre z_0

Thus, $z_2 - z_0 = (z_1 - z_0)\,e^{i\pi/2}$

or $\quad z_2 - 1 = (2 + i\sqrt{3} - 1)\,i \Rightarrow z_2 = i - \sqrt{3} + 1$

$z_2 = (1-\sqrt{3}) + i$

Again rotating z_2 by 90° about z_0 we get

$z_3 - z_0 = (z_2 - z_0)\,i$

$\Rightarrow \quad z_3 - 1 = [(1-\sqrt{3}) + i - 1]\,i = -\sqrt{3}i - 1 \Rightarrow z_3 = -i\sqrt{3}$

and similarly $1 = (-i\sqrt{3} - 1)\,i = \sqrt{3} - i$

$\Rightarrow \quad z_4 = (\sqrt{3}+1) - i$

Thus the remaining vertices are

$(1-\sqrt{3}) + i, -i\sqrt{3}, (\sqrt{3}+1) - i$

F. Match the Following

1. $z \ne 0$ Let $z = a + ib$

Re (z) = 0 $\Rightarrow z = ib \Rightarrow z^2 = -b^2$

$\therefore$ *Im* $(z)^2 = 0$

$\therefore$ (A) corresponds to (q)

Arg $z = \dfrac{\pi}{4} \Rightarrow a = b \Rightarrow z = a + ia$

$z^2 = a^2 - a^2 + 2ia^2; \qquad z^2 = 2ia^2 \Rightarrow \text{Re}\,(z)^2 = 0$

$\therefore$ (B) corresponds to (p).

2. (A) $\to$ (q, r) $\big|z - i|z|\big| = \big|z + i|z|\big|$

$\Rightarrow z$ is equidistant from two points $(0, |z|)$ and $(0, -|z|)$ which lie on imaginary axis.

$\therefore z$ must lie on real axis $\Rightarrow$ Im(z) = 0 also $|I_m(z)| \le 1$

(B) $\to$ p

Sum of distances of z from two fined points (–4, 0) and (4, 0) is 10 which is greater than 8.

$\therefore z$ traces an ellipse with $2a = 10$ and $2ae = 8$

$\Rightarrow e = 4/5$

(C) $\rightarrow$ (p, s, t)

Let $\omega = 2(\cos\theta + i\sin\theta)$

then $z = \omega - \frac{1}{\omega} = 2(\cos\theta + i\sin\theta) - \frac{1}{2}(\cos\theta - i\sin\theta)$

$\Rightarrow\ x + iy = \frac{3}{2}\cos\theta + i\frac{5}{2}\sin\theta$

Here $|z| = \sqrt{\frac{9+25}{4}} = \sqrt{\frac{34}{4}} \le 3$ and $|R_e(z)| \le 2$

Also $x = \frac{3}{2}\cos\theta, y = \frac{5}{2}\sin\theta \ \Rightarrow\ \frac{4x^2}{9} + \frac{4y^2}{25} = 1$

Which is an ellipse with $e = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

(D) $\rightarrow$ (q, r, s, t)

Let $\omega = \cos\theta + i\sin\theta$ then $z = 2\cos\theta \Rightarrow \text{Im}\, z = 0$

Also $|z| \le 3$ and $|\text{Im}(z)| \le 1, |\text{R}_e(z)| \le 2$

3. **(c)** (P) $\rightarrow$ (1) : $z_k = \cos\frac{2k\pi}{10} + i\sin\frac{2k\pi}{10}$, $k = 1$ to 9

$\therefore\ z_k = e^{i\frac{2k\pi}{10}}$

Now $z_k.z_j = 1 \Rightarrow z_j = \frac{1}{z_k} = e^{-i\frac{2k\pi}{10}} = \overline{z_k}$

We know if z_k is 10th root of unity so will be $\overline{z}_k$.

$\therefore$ For every z_k, there exist $z_i = \overline{z}_k$

Such that $z_k.z_j = z_k.\overline{z}_k = 1$

Hence the statement is true.

(Q) $\rightarrow$ (2) $z_1 = z_k \ \Rightarrow z = \frac{z_k}{z_1}$ for $z_1 \ne 0$

$\therefore$ We can always find a solution to $z_1.z = z_k$

Hence the statement is false.

(R) $\rightarrow$ (3) : We know $z^{10} - 1 = (z-1)(z-z_1)....(z-z_9)$

$\Rightarrow (z-z_1)(z-z_2).....(z-z_9) = \frac{z^{10}-1}{z-1}$

$= 1 + z + z^2 + ... z^9$

For z = 1 we get

$(1-z_1)(1-z_2).....(1-z_9) = 10$

$\therefore\ \frac{|1-z_1||1-z_2|.....|1-z_9|}{10} = 1$

(S) $\rightarrow$ (4) : $1, Z_1, Z_2, ... Z_9$ are 10th roots of unity.

$\therefore Z^{10} - 1 = 0$

From equation $1 + Z_1 + Z_2 + ... + Z_9 = 0$

$\text{Re}(1) + \text{Re}(Z_1) + \text{Re}(Z_2) + ... + \text{Re}(Z_9) = 0$

$\Rightarrow \text{Re}(Z_1) + \text{Re}(Z_2) + ... \text{Re}(Z_9) = -1$

$\Rightarrow \sum_{K=1}^{9} \cos\frac{2k\pi}{10} = -1 \Rightarrow 1 - \sum_{K=1}^{9} \cos\frac{2k\pi}{10} = 2$

Hence (c) is the correct option.

G. Comprehension Based Questions

For (Q. 1 - 3)

We have $A = \{z : \text{Im}(z) \ge 1\} = \{(x, y) : y \ge 1\}$

Clearly A is the set of all points lying on or above the line $y = 1$ in cartesian plane.

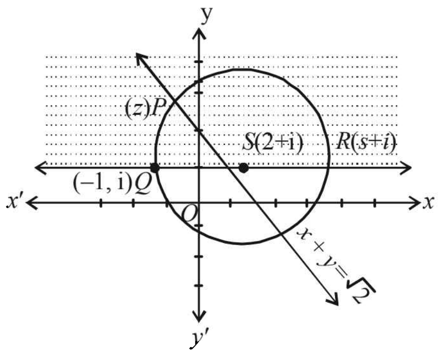
$B = \{z : |z - 2 - i| = 3\} = \{(x, y) : (x-2)^2 + (y-1)^2 = 9\}$

$\Rightarrow B$ is the set of all points lying on the boundary of the circle with centre (2, 1) and radius 3.

$C = \{z : \text{Re}[(1-i)z] = \sqrt{2}\} = \{(x, y) : x + y = \sqrt{2}\}$

$\Rightarrow$ C is the set of all points lying on the straight line represented by $x + y = \sqrt{2}$.

Graphically, the three sets are represented as shown below :



1. **(b)** From graph $A \cap B \cap C$ consists of only one point P [the common point of the region $y \ge 1, (x-2)^2 + (y-1)^2 = 9$ and $x + y = \sqrt{2}$] $\quad \therefore n(A \cap B \cap C) = 1$

2. **(c)** As z is a point of $A \cap B \cap C \Rightarrow z$ represents the point P

$\therefore |z + 1 - i|^2 + |z - 5 - i|^2 \Rightarrow |z - (-1 + i)|^2 + |z - (5 + i)|^2$

$\Rightarrow PQ^2 + PR^2 = QR^2 = 6^2 = 36$

which lies between 35 and 39

$\therefore$ (c) is correct option.

3. **(d)** Given that $|w - 2 - i| < 3$

$\Rightarrow$ Distance between w and $2 + i$ i.e. S is smaller than 3.

$\Rightarrow w$ is a point lying inside the circle with centre S and radius 3.

$\Rightarrow$ Distance between z (i.e. the point P) and w should be smaller than 6 (the diameter of the circle)

i.e. $|z-w|<6$

But we know that $||z|-|w|| < |z-w|$

$\Rightarrow ||z|-|w|| < 6 \Rightarrow -6 < |z|-|w| < 6$

$-3 < |z|-|w|+3 < 9$

For (Q. 4 & 5)

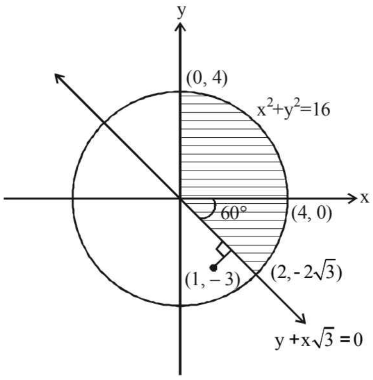
$S_1 : x^2+y^2 < 16$

$S_2 : \text{Im}\left[\dfrac{(x-1)+i(y+\sqrt{3})}{1-i\sqrt{3}}\right] > 0$

$\Rightarrow \sqrt{3}(x-1)+(y+\sqrt{3}) > 0 \Rightarrow y+\sqrt{3}x > 0$

$S_3 : x > 0$

Then S : $S_1 \cap S_2 \cap S_3$ is as shown in the figure given below.



4. **(b)** Area of shaded region

$$=\frac{\pi}{4}\times 4^2+\frac{\pi\times 4^2\times 60°}{360°}=4\pi+\frac{8\pi}{3}=\frac{20\pi}{3}$$

5. **(c)** $\min_{z\in s}|1-3i-z| =$ min distance between z and $(1,-3)$

Clearly (from figure) minimum distance between $z \in s$ and $(1,-3)$ from line $y+x\sqrt{3}=0$ i.e. $\left|\dfrac{\sqrt{3}-3}{\sqrt{3}+1}\right| = \dfrac{3-\sqrt{3}}{2}$

I. Integer Value Correct Type

1. **(5)**

Given $|z-3-2i| \le 2$

which represents a circular region with centre (3, 2) and radius 2.

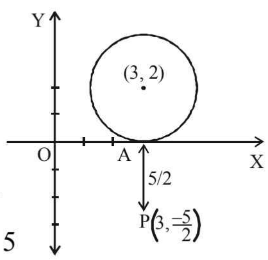
Now $|2z-6+5i| = 2\left|z-\left(3-\dfrac{5}{2}i\right)\right|$

$= 2 \times$ distance of z from P

(where Z lies in or on the circle)

Also min distance of z from $P = \dfrac{5}{2}$

$\therefore$ Minimum value of $|2z-6+5i| = 5$



2. **(3)**

The expression may not attain integral value for all a, b, c. If we consider a = b = c then

$x = 3a, y = a(1+\omega+\omega^2) = a(1+i\sqrt{3})$

$Z = a(1+\omega^2+\omega) = a(1+i\sqrt{3})$

$\Rightarrow |x|^2+|y|^2+|z|^2 = 9|a|^2+4|a|^2+4|a|^2 = 17|a|^2$

$\Rightarrow \dfrac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2} = \dfrac{17}{3}$ (which is not an integer)

Note : However if $\omega = e^{i(2\frac{\pi}{3})}$, then the value of expression can be evaluated as follows

$$\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2} = \frac{x\bar{x}+y\bar{y}+z\bar{z}}{|a|^2+|b|^2+|c|^2}$$

$$= \frac{(a+b+c)(\bar{a}+\bar{b}+\bar{c})+(a+b\omega+c\omega^2)(\bar{a}+\bar{b}\omega^2+\bar{c}\omega)+(a+b\omega^2+c\omega)(\bar{a}+\bar{b}\omega+\bar{c}\omega^2)}{|a|^2+|b|^2+|c|^2}$$

$$= \frac{3|a|^2+3|b|^2+3|c|^2+(a\bar{b}+\bar{a}b+b\bar{c}+\bar{b}c+a\bar{c}+\bar{a}c)(1+\omega+\omega^2)}{|a|^2+|b|^2+|c|^2}$$

$= 3 \qquad (\because 1+\omega+\omega^2 = 0)$

3. **(4)** $\alpha_k = \cos\dfrac{k\pi}{7}+i\sin\dfrac{k\pi}{7} = e^{\frac{i\pi k}{7}}$

$$\alpha_{k+1}-\alpha_k = e^{\frac{i\pi(k+1)}{7}} - e^{\frac{i\pi k}{7}} = e^{\frac{i\pi k}{7}}(e^{i\pi/7}-1)$$

$$|\alpha_{k+1}-\alpha_k| = |e^{i\pi/7}-1|$$

$$\Rightarrow \sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k| = 12|e^{i\pi/7}-1|$$

Similarly $\displaystyle\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}| = 3|e^{i\pi/7}-1|$

$$\therefore \frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|} = 4$$

Section-B JEE Main/ AIEEE

1. (b) Let $|z|=|\omega|=r$

$\therefore z=re^{i\theta},\ \omega=re^{i\phi}$ where $\theta+\phi=\pi$.

$\therefore z=re^{i(\pi-\phi)}=re^{i\pi}.e^{-i\phi}=-re^{-i\phi}=-\bar{\omega}.[\because \bar{\omega}=re^{-i\phi}]$

2. (c) Given $|z-4|<|z-2|$ Let $z=x+iy$

$\Rightarrow |(x-4)+iy)|<|(x-2)+iy|$

$\Rightarrow (x-4)^2+y^2<(x-2)^2+y^2$

$\Rightarrow x^2-8x+16<x^2-4x+4 \Rightarrow 12<4x$

$\Rightarrow x>3 \Rightarrow \text{Re}(z)>3$

3. (b) Let the circle be $|z-z_0|=r$. Then according to given conditions $|z_0-z_1|=r+a$ and $|z_0-z_2|=r+b$. Eliminating r, we get $|z_0-z_1|-|z_0-z_2|=a-b$.

$\therefore$ Locus of centre z_0 is $|z-z_1|-|z-z_2|=a-b$, which represents a hyperbola.

4. (a) $|\bar{z}\omega|=|\bar{z}||\omega|=|z||\omega|=|z\omega|=1$

$\text{Arg}(\bar{z}\omega)=\arg(\bar{z})+\arg(\omega)=-\arg(z)+\arg\omega$

$=-\frac{\pi}{2}\quad \therefore \bar{z}\omega=-1$

5. (d) $z^2+az+b=0$; $z_1+z_2=-a\ \&\ z_1z_2=b$

$0, z_1, z_2$ form an equilateral Δ

$\therefore 0^2+z_1^2+z_2^2=0.z_1+z_1.z_2+z_2.0$

(for an equilateral triangle,

$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$)

$\Rightarrow z_1^2+z_2^2=z_1z_2 \Rightarrow (z_1+z_2)^2=3z_1z_2 \quad \therefore a^2=3b$

6. (b) $\left(\frac{1+i}{1-i}\right)^x=1 \Rightarrow \left[\frac{(1+i)^2}{1-i^2}\right]^x=1$

$\left(\frac{1+i^2+2i}{1+1}\right)^x=1 \Rightarrow (i)^x=1;\ \therefore x=4n;\ n\in I^+$

7. (c) $\arg zw=\pi \Rightarrow \arg z+\arg w=\pi$...(1)

$\bar{z}+i\bar{w}=0 \Rightarrow \bar{z}=-i\bar{w}$

$\therefore z=iw \Rightarrow \arg z=\frac{\pi}{2}+\arg w$

$\Rightarrow \arg z=\frac{\pi}{2}+\pi-\arg z$ (from(1)) $\therefore \arg z=\frac{3\pi}{4}$

8. (a) $z^{\frac{1}{3}}=p+iq \Rightarrow z=p^3+(iq)^3+3p(iq)(p+iq)$

$\Rightarrow x-iy=p^3-3pq^2+i(3p^2q-q^3)$

$\therefore x=p^3-3pq^2 \Rightarrow \frac{x}{p}=p^2-3q^2$

$y=q^3-3p^2q \Rightarrow \frac{y}{q}=q^2-3p^2$

$\therefore \frac{x}{p}+\frac{y}{q}=-2p^2-2q^2\ \therefore \left(\frac{x}{p}+\frac{y}{q}\right)\Big/(p^2+q^2)=-2$

9. (b) $|z^2-1|=|z|^2+1 \Rightarrow |z^2-1|^2=(z\bar{z}+1)^2$

$\Rightarrow (z^2-1)(\bar{z}^2-1)=(z\bar{z}+1)^2$

$\Rightarrow z^2\bar{z}^2-z^2-\bar{z}^2+1=z^2\bar{z}^2+2z\bar{z}+1$

$\Rightarrow z^2+2z\bar{z}+\bar{z}^2=0 \Rightarrow (z+\bar{z})^2=0 \Rightarrow z=-\bar{z}$

$\Rightarrow z$ is purely imaginary

10. (c) $(x-1)^3+8=0 \Rightarrow (x-1)=(-2)\,(1)^{1/3}$

$\Rightarrow x-1=-2$ or -2ω or $-2\omega^2$

or $x=-1$ or $1-2\omega$ or $1-2\omega^2$.

11. (c) $|z_1+z_2|=|z_1|+|z_2| \Rightarrow z_1$ and z_2 are collinear and are to the same side of origin; hence $\arg z_1-\arg z_2=0$.

12. (c) As given $w=\frac{z}{z-\frac{1}{3}i} \Rightarrow |w|=\frac{|z|}{|z-\frac{1}{3}i|}=1$

$\Rightarrow |z|=\left|z-\frac{1}{3}i\right|$

$\Rightarrow$ distance of z from origin and point $\left(0,\frac{1}{3}\right)$ is same hence z lies on bisector of the line joining points (0, 0) and (0, 1/3).

Hence z lies on a straight line.

13. (d)

$$\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11}+i\cos\frac{2k\pi}{11}\right)=i\sum_{k=1}^{10}\left(\cos\frac{2k\pi}{11}-i\sin\frac{2k\pi}{11}\right)$$

$$=i\sum_{k=1}^{10}e^{-\frac{2k\pi}{11}i}=i\left\{\sum_{k=0}^{10}e^{-\frac{2k\pi}{11}i}-1\right\}$$

$$=i\left[1+e^{-\frac{2\pi}{11}i}+e^{-\frac{4\pi}{11}i}+\text{....11 terms}\right]-i$$

$$=i\left[\frac{1-\left(e^{-\frac{2\pi}{11}}\right)^{11}}{1-e^{-\frac{2\pi}{11}i}}\right]-i=i\left[\frac{1-e^{-2\pi i}}{1-e^{-\frac{2\pi}{11}i}}\right]-i$$

$=i\times 0-i \quad [\because e^{-2\pi i}=1]=-i$

14. (d) $z^2+z+1=0 \Rightarrow z=\omega$ or ω^2

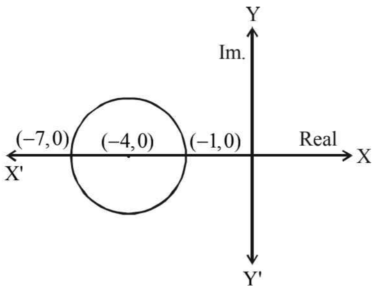
So, $z+\frac{1}{z}=\omega+\omega^2=-1$

$z^2+\frac{1}{z^2}=\omega^2+\omega=-1,\ z^3+\frac{1}{z^3}=\omega^3+\omega^3=2$

$z^4 + \frac{1}{z^4} = -1,\ z^5 + \frac{1}{z^5} = -1$ and $z^6 + \frac{1}{z^6} = 2$

$\therefore$ The given sum $= 1 + 1 + 4 + 1 + 1 + 4 = 12$

15. (a) z lies on or inside the circle with centre $(-4, 0)$ and radius 3 units.



From the Argand diagram maximum value of $|z + 1|$ is 6

16. (c) $\left(\frac{1}{i-1}\right) = \frac{1}{-i-1} = \frac{-1}{i+1}$

17. (d) Given $S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$

$\because x \neq x + 1$ for any $x \in (0, 2) \Rightarrow (x, x) \notin S$

$\therefore S$ is not reflexive.

Hence S in not an equivalence relation.

Also $T = \{x, y) : x - y$ is an integer$\}$

$\because x - x = 0$ is an integer $\forall x \in R$

$\therefore T$ is reflexive.

If $x - y$ is an integer then $y - x$ is also an integer

$\therefore T$ is symmetric

If $x - y$ is an integer and $y - z$ is an integer then $(x - y) + (y - z) = x - z$ is also an integer.

$\therefore T$ is transitive

Hence T is an equivalence relation.

18. (a) Let $z = x + iy$

$|z - 1| = |z + 1| (x-1)^2 + y^2 = (x+1)^2 + y^2$

$\Rightarrow \text{Re}\, z = 0 \quad \Rightarrow x = 0$

$|z - 1| = |z - i| (x-1)^2 + y^2 = x^2 + (y-1)^2$

$\Rightarrow x = y$

$|z + 1| = |z - i| (x+1)^2 + y^2 = x^2 + (y-1)^2$

Only $(0, 0)$ will satisfy all conditions.

$\Rightarrow$ Number of complex number $z = 1$

19. (c) $\because$ Real part of roots is 1

Let roots are $1 + pi,\ 1 + q$

$\therefore$ sum of roots $= 1 + pi + 1 + qi = -\alpha$ which is real

$\Rightarrow q = -p$ or root are

$1 + pi$ and $1 - pi$ product of roots $= 1 + p^2 = \beta \in (1, \infty)$

$p \neq 0$ as roots are distinct.

20. (a) $(1 + \omega)^7 = A + B\omega$; $\quad (-\omega^2)^7 = A + B\omega$

$-\omega^2 = A + B\omega$; $\quad 1 + \omega = A + B\omega$

$\Rightarrow A = 1, B = 1.$

21. (a) Let $z = x + iy \ \therefore\ z^2 = x^2 - y^2 + 2ixy$

Now $\frac{z^2}{z-1}$ is real $\Rightarrow \text{Im}\left(\frac{z^2}{z-1}\right) = 0$

$\Rightarrow \text{Im}\left(\frac{x^2 - y^2 + 2ixy}{(x-1) + iy}\right) = 0$

$\Rightarrow \text{Im}\,[(x^2 - y^2 + 2ixy)(x - 1) - iy)] = 0$

$\Rightarrow 2xy(x - 1) - y(x^2 - y^2) = 0$

$\Rightarrow y(x^2 + y^2 - 2x) = 0 \Rightarrow y = 0;\ x^2 + y^2 - 2x = 0$

$\therefore z$ lies either on real axis or on a circle through origin.

22. (c) Given $|z| = 1$, $\arg z = \theta$

As we know, $\bar{z} = \frac{1}{z}$

$\therefore \arg\left(\frac{1+z}{1+\bar{z}}\right) = \arg\left(\frac{1+z}{1+\frac{1}{z}}\right) = \arg(z) = \theta.$

23. (b) We know minimum value of $|Z_1 + Z_2|$ is $||Z_1| - |Z_2||$

Thus minimum value of $\left|Z + \frac{1}{2}\right|$ is $\left||Z| - \frac{1}{2}\right|$

$\leq \left|Z + \frac{1}{2}\right| \leq |Z| + \frac{1}{2}$

Since, $|Z| \geq 2$ therefore $2 - \frac{1}{2} < \left|Z + \frac{1}{2}\right| < 2 + \frac{1}{2}$

$\Rightarrow \frac{3}{2} < \left|Z + \frac{1}{2}\right| < \frac{5}{2}$

24. (a) $\left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1 \Rightarrow |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

$\Rightarrow (z_1 - 2z_2)\overline{(z_1 - 2z_2)} = (2 - z_1\bar{z}_2)\overline{(2 - z_1\bar{z}_2)}$

$\Rightarrow (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$

$\Rightarrow (z_1\bar{z}_1) - 2z_1\bar{z}_2 - 2\bar{z}_1 z_2 + 4z_2\bar{z}_2$

$= 4 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2 + z_1\bar{z}_1 z_2\bar{z}_2$

$\Rightarrow |z_1|^2 + 4|z_2|^2 = 4 + |z_1|^2 |z_2|^2$

$\Rightarrow |z_1|^2 + 4|z_2|^2 - 4 - |z_1|^2 |z_2|^2 = 0$

$\left(|z_1|^2 - 4\right)\left(1 - |z_2|^2\right) = 0$

$\because |z_2| \neq 1 \ \therefore |z_1|^2 = 4 \Rightarrow |z_1| = 2$

$\Rightarrow$ Point z_1 lies on circle of radius 2.

25. (b) Rationalizing the given expression

$\frac{(2 + 3i\sin\theta)(1 + 2i\sin\theta)}{1 + 4\sin^2\theta}$

For the given expression to be purely imaginary, real part of the above expression should be equal to zero.

$\Rightarrow \frac{2 - 6\sin^2\theta}{1 + 4\sin^2\theta} = 0 \quad \Rightarrow \sin^2\theta = \frac{1}{3} \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}}$